Lecture Sheet on Numerical Analysis

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Basic Information

- Course Credit: 03
- CIE Marks: 90
- SEE marks: 60
- Semester End Exam (SEE): 3 hours

Course Learning Outcomes

01 Student will acquire proficiency in fundamental concepts and apply it to solve different problems.

02 Develop advanced analytical and problem-solving skills essential for tackling complex computational challenges.

03 Demonstrate mastery several mathematical proofs and apply the method to solve problems.

04 Apply theoretical knowledge to effectively solve practical problems.

SL.	Content of Courses	Hrs	CLO’s
1	Solving non-linear equations such as Bi- Section, False Position, Newton-Raphson, Fixed Point Iteration, and Bairstow’s Method. Solving system of linear equations solutions including Triangular systems and back substitution.	10	CLO1, CLO2
2	LU-factorization, Pivoting, Interpolation and approximation techniques like Newton’s forward and backward interpolation, Divided differences formula, Lagrange polynomial.	8	CLO1, CLO4
3	Richardson’s extrapolation, and numerical differentiation techniques	8	CLO1, CLO3
4	Trapezoidal rule, Simpson’s rule (1/3 rule and 3/8 rule), and Romberg’s integration, Euler’s method, Modified Euler’s method	8	CLO2, CLO4

Step-by-step Progression

- Step 1: Lecture
- Step 2: Home Work
- Step 3: Practice
- Step 4: Present/Solve

Marks Distribution

45

Assignment, Quiz, Attendence

45

Mid Term

60

Final

Assessment Pattern

Blooms Category	Test (Out of 45)	Assignments (15)	Quiz (15)	Co-curricular Activities (15)
Remember	05		5	Attendance 15
Understand	05			
Apply	10			
Analysis	8	7	10	
Evaluate	7	8		
Create	10			

Semester End Examination (SEE 60)

Blooms Category	Test (Out of 60)
Remember	10
Understand	10
Apply	10
Analysis	10
Evaluate	10
Create	10

Assessment Pattern (SEE)

Test Score

Remember	10
Understand	10
Apply	10
Analysis	10
Evaluate	10
Create	10

How to Get Best Result?

Stay consistent with a study routine and focus on understanding concepts instead of memorizing. Practice regularly and discuss with friends.

Attend Class

Group Study

Practice

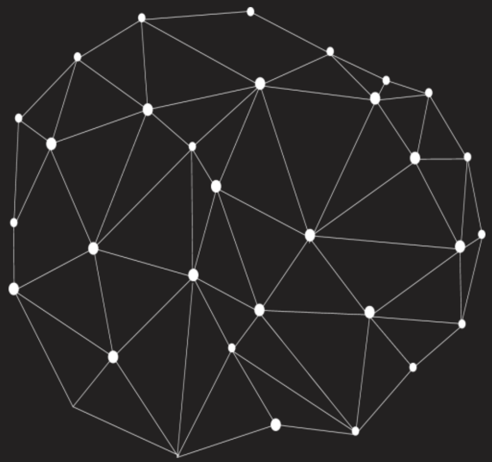
Understand

Apply

Weekly Plan

W S

Week 1							
Week 2							
Week 3							
... ...							
Week 17							



Course Plan Specific Content, CLOs, Teaching Learning and Assessment Strategy mapped with CLOs.

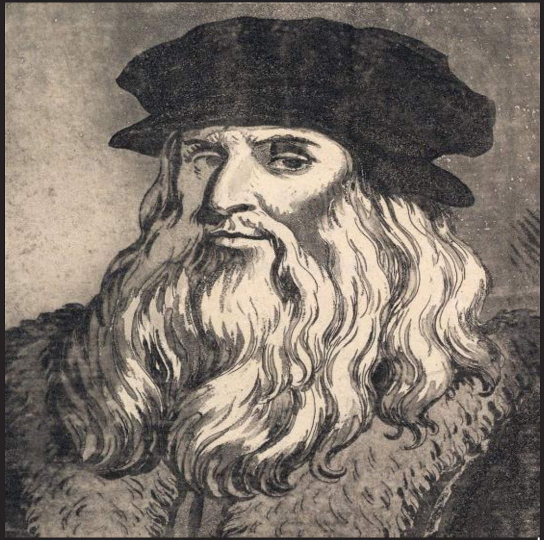
Week No.	Task Heading	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
1	Numerical Integration	Trapezoidal rule	Lecture, Discussion	Presentation, Quiz	CLO2, CLO4
2	Numerical Integration	Simpson's rule 1/3 rule	Lecture, Discussion	Quiz, Written Assignment, Oral Presentation	CLO2, CLO4
3	Numerical differentiation	Simpson's rule 3/8 rule	Lecture, Discussion	Oral Presentation, Group Assignment	CLO2, CLO4
4	Numerical differentiation	Richardson's extrapolation	Lecture, Discussion	Oral Presentation	CLO1, CLO3
5	Solution of Non-linear Equations	Bisection method	Discussion, Oral Presentation	Written Assignment	CLO1, CLO2
6	Solution of Non-linear Equations	False Position method	Discussion, Oral Presentation	Written Assignment	CLO1, CLO2

7	Solution of Non-linear Equations	Newton- Raphson method	Oral Presentation	Oral Presentation	CLO1, CLO2
8	Solution of Non-linear Equations	Fixed Point Iteration	Group Work	Group Assignment	CLO1, CLO2
9	Solution of system of Linear equations	Triangular systems and back substitution, Forward and Backward Substitution	Group Work	Quiz, Written Assignment	CLO1, CLO2
10	Solution of system of Linear equations	Gauss elimination method	Lecture, Discussion	Oral Presentation, Quiz	CLO1, CLO2
11	Solution of system of Linear equations	LU-factorization	Discussion, Oral Presentation	Group Assignment, Quiz	CLO1, CLO4
12	Solution of system of Linear equations	Pivoting	Oral Presentation	Presentation, Written Assignment	CLO1, CLO2

13	Interpolation and Approximation	Newton's forward and backward interpolation	Oral Presentation	Quiz, Presentation	CLO1, CLO2,
14	Interpolation and Approximation	Newton's forward and backward interpolation	Group Work	Written Assignment, Oral Presentation	CLO1, CLO2, CLO3
15	Interpolation and Approximation	Divided differences formula	Discussion, Oral Presentation	Group Assignment, Presentation	CLO1, CLO2
16	Interpolation and Approximation	Lagrangian interpolation	Discussion, Oral Presentation	Quiz, Group Assignment	CLO1, CLO2
17	Interpolation and Approximation	Lagrangian interpolation	Oral Presentation	Written Assignment, Quiz	CLO1, CLO2

“No human investigation can be called real science if it cannot be demonstrated mathematically.”

—Leonardo Da Vinci



Reference Book

1. Numerical Analysis" by Richard L. Burden and J. Douglas Faires

A classic textbook offering a detailed understanding of numerical techniques, including their derivations, applications, and error analysis.

2. Numerical Methods for Engineers" by Steven C. Chapra and Raymond P. Canale

A practical guide focused on numerical methods with applications in engineering problems.

"Dear students, let’s turn numbers into adventures and discover how fun and exciting math can truly be!"

Week 1

Topic: Trapezoidal Rule Page: 05- 06

Numerical Analysis Overview

- What is Numerical analysis?
- Applications
- Exact and approximate solution

Integration Method

Trapezoidal

Simpson's 1/3 Rule

Simpson's 3/8 Rule

Trapeziodal Rule

The trapezoidal rule is a numerical method used to approximate the definite integral of a function. It divides the area under a curve into trapezoids, calculates their areas, and sums them up for an estimate of the integral.

The Trapezoidal Rule: A Stunning Mathematical Approach

Welcome to the world of numerical integration! Today, we'll dive into the Trapezoidal Rule, a powerful method for approximating the area under a curve.

Understanding the Trapezoidal Method

The Basics

The Trapezoidal Rule estimates the area by dividing the region under the curve into trapezoids.

Formula

The formula for the area of each trapezoid is (base1 + base2) / 2 * height.

Visualizing the Trapezoidal Approximation

1

Step 1

Divide the region under the curve into equal subintervals.

2

Step 2

Connect the endpoints of each subinterval to form trapezoids.

3

Step 3

Calculate the area of each trapezoid and sum them.

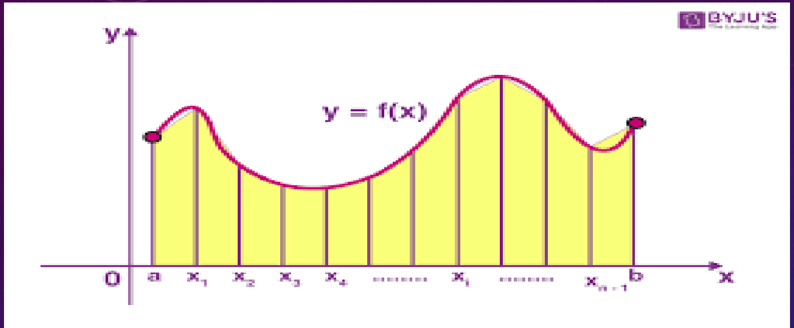
Comparing Trapezoidal to Other Integration Techniques

Advantages

Simple to understand and implement. Relatively accurate for smooth curves.

Disadvantages

Accuracy depends on the number of subintervals. Can be less accurate than other methods for complex curves.

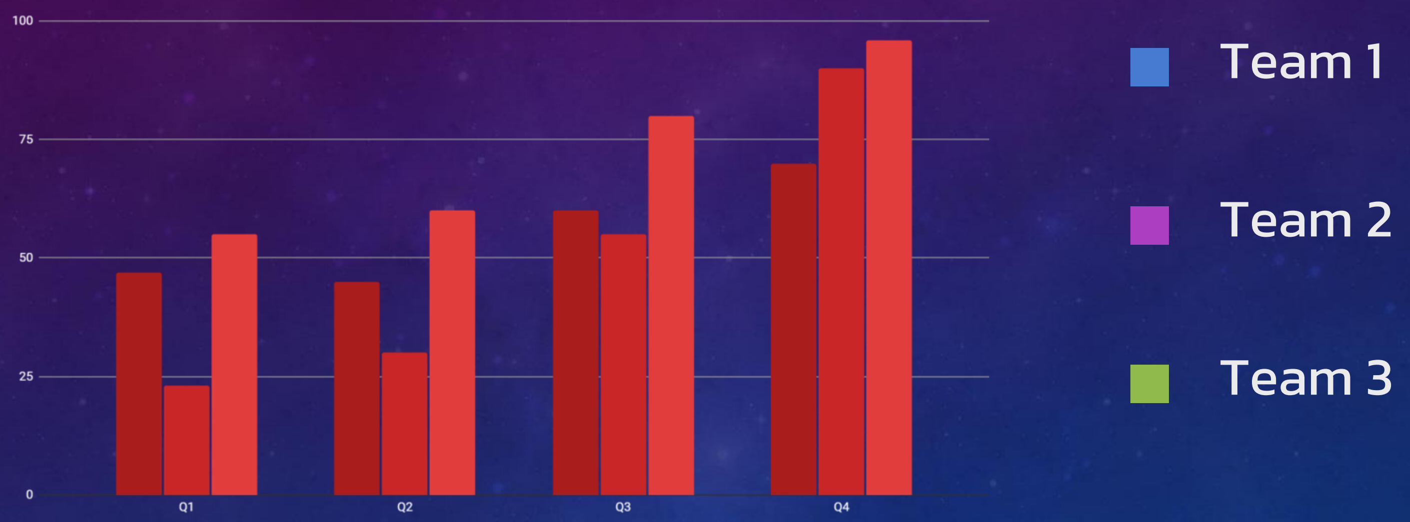


Question: Compute the integral $\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$ by using **Trapezoidal rule**.

Solution: Here $a = 0$ and $b = 1$, we shall divide the interval into 10 equals parts. Hence $h = \frac{1-0}{10} = 0.1$ Now we find the values of $y = \frac{1}{\sqrt{1+x^2}}$ for each point of subdivision in the following table.

x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = 0.995037$
$x_2 = 0.2$	$y_2 = 0.980580$
$x_3 = 0.3$	$y_3 = 0.957826$
$x_4 = 0.4$	$y_4 = 0.928476$
$x_5 = 0.5$	$y_5 = 0.894427$
$x_6 = 0.6$	$y_6 = 0.857493$
$x_7 = 0.7$	$y_7 = 0.819232$
$x_8 = 0.8$	$y_8 = 0.780869$
$x_9 = 0.9$	$y_9 = 0.743294$
$x_{10} = 1.0$	$y_{10} = 0.707107$

TIME FOR GROUP WORK!

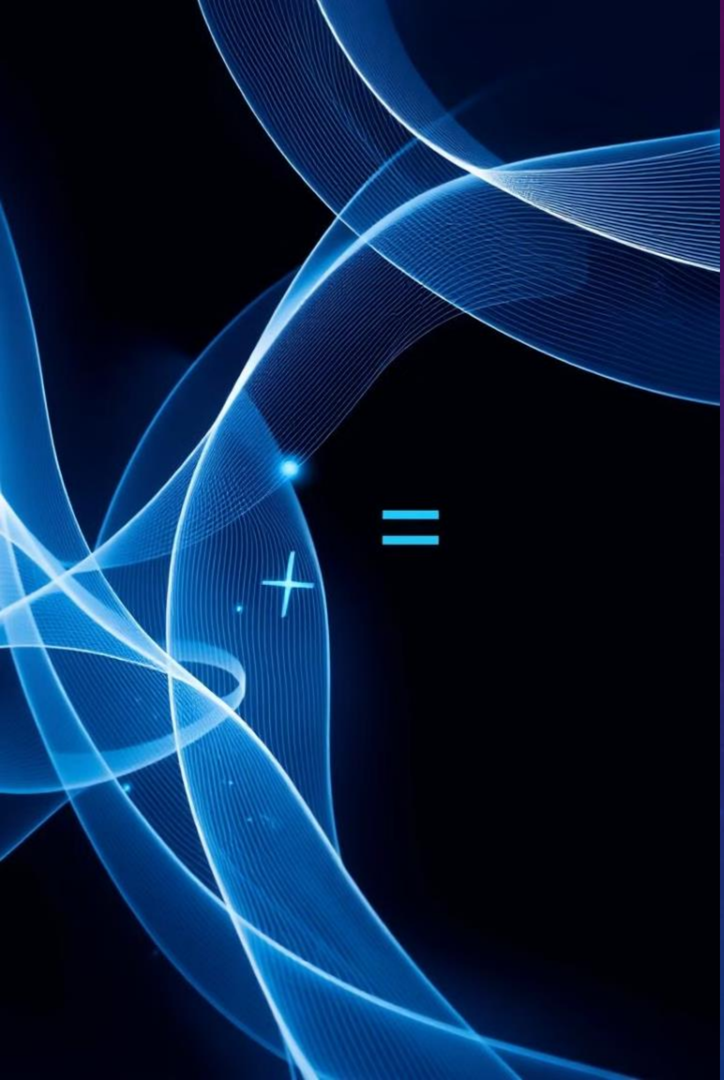


Week 2

Topic: Simpson’s 1/3 Rule Page: 06- 07

"The definition of a good mathematical problem is the mathematics it generates rather than the problem itself".

—Andrew Wiles



Simpson's 1/3 and 3/8 Rules

This presentation explores Simpson's 1/3 and 3/8 rules for numerical integration. We'll delve into their derivation, accuracy, and application with stunning visuals. We'll also discuss limitations and considerations.

Introduction to Numerical Integration

Background

Numerical integration approximates the definite integral of a function. It's crucial when analytical methods fail.

Applications

Numerical integration is used in various fields, including engineering, physics, and finance.

Derivation of the Simpson's 1/3 Rule

1. **Parabolic Approximation**

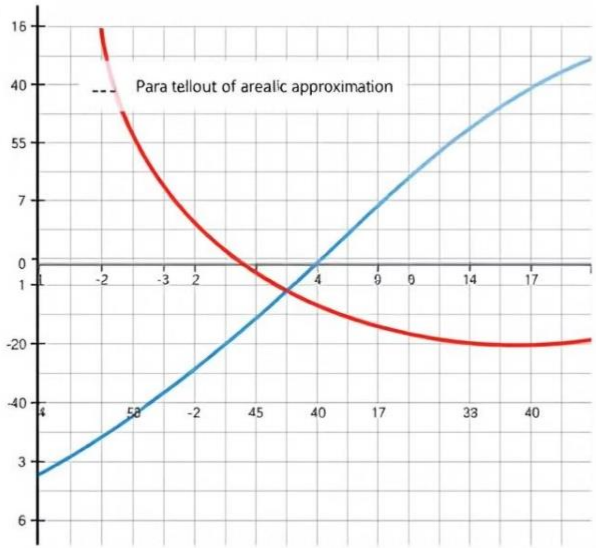
   Simpson's 1/3 rule uses a parabolic approximation to estimate the area under the curve.

2. **Three Points**

   It requires three equally spaced points: two endpoints and one midpoint.

3. **Formula**

   The formula integrates the parabolic function to approximate the area.



Proof of the Accuracy of Simpson's 1/3 Rule

1

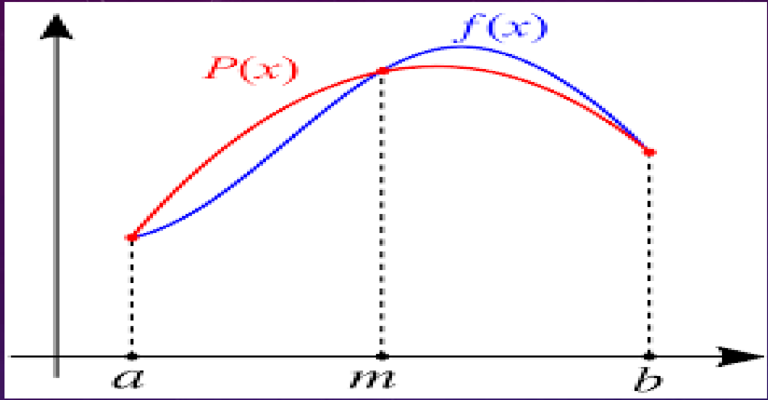
Error Term

The error term in Simpson's 1/3 rule is proportional to the fourth derivative of the function.

2

Higher Accuracy

For functions with continuous fourth derivatives, the rule provides higher accuracy than the Trapezoidal rule.



<u>Question</u>: Compute the integral $\int_0^6 \frac{1}{(1+x)^2}\, dx$

by using **Simpson's $\frac{1}{3}$ rule.**

Solution: Here $a = 0$ and $b = 6$, we shall divide the interval into 6 equals parts.

Hence $h = \frac{6-0}{6} = 1$ Now we find the values of $y = \frac{1}{(1+x)^2}$ for each point of subdivision in the following table.

x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 1$	$y_1 = 0.25000$
$x_2 = 2$	$y_2 = 0.11111$
$x_3 = 3$	$y_3 = 0.06250$

$x_4 = 4$	$y_4 = 0.04000$
$x_5 = 5$	$y_5 = 0.02778$
$x_6 = 6$	$y_6 = 0.02041$

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^6 \frac{1}{(1+x)^2} dx$$

$$= \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5)$$

$$+ 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.02041)$$

$$+ 4(0.25000 + 0.06250$$

$$+ 0.02778)$$

$$+ 2(0.11111 + 0.04000)]$$

$$= 0.89458$$

Week 3

Topic: Simpson's 3/8 Rule

Page: 7-8

Derivation of the Simpson's 3/8 Rule

1. **Cubic Approximation**

   Simpson's 3/8 rule uses a cubic approximation to estimate the area.

2. **Four Points**

   It requires four equally spaced points: two endpoints and two interior points.

3. **Formula**

   The formula integrates the cubic function to approximate the area.

Proof of the Accuracy of Simpson's 3/8 Rule

1

Error Term

The error term in Simpson's 3/8 rule is proportional to the fifth derivative of the function.

2

Higher Accuracy (Often)

For functions with continuous fifth derivatives, the rule provides higher accuracy than the 1/3 rule.

Comparison of the Two Rules

Simpson's 1/3

Uses a parabolic approximation. Requires three points. Error is proportional to the fourth derivative.

Simpson's 3/8

Uses a cubic approximation. Requires four points. Error is proportional to the fifth derivative.

Example Application with Stunning Visuals

1

Function

Let's integrate f(x) = x from 0 to 2 using both rules.

2

Result

The Simpson's 1/3 and 3/8 rules provide accurate approximations of the integral's value.

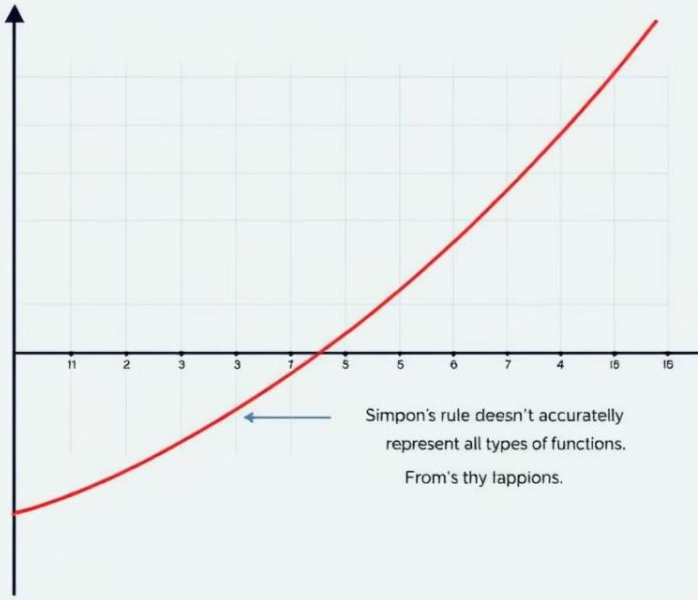
Limitations and Considerations

Oscillatory Functions

Simpson's rules may be less accurate for highly oscillatory functions.

Discontinuities

They may not be suitable for functions with discontinuities within the integration interval.



Conclusion and Key Takeaways

Simpson's 1/3 and 3/8 rules are powerful numerical integration techniques. They offer higher accuracy than simpler methods. Understanding their derivations, applications, and limitations is crucial for effective use.

$x = 4.2$ 1	y_1 $= 1.43508$
$x = 4.4$ 2	y_2 $= 1.48160$
$x = 4.6$ 3	y_3 $= 1.52606$
$x = 4.8$ 4	y_4 $= 1.56862$
$x = 5.0$ 5	y_5 $= 1.60944$
$x = 5.2$ 6	y_6 $= 1.64866$

$$= \frac{0.2}{3}[(1.38629 + 1.64866)$$

$$+ 2(1.43508 + 1.48160 + 1.52606$$

$$+ 1.56862 + 1.60944)]$$

$$= 1.82766$$

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_{4}^{5.2} \ln x \, dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$x_2 = 0.6$	$y_2 = 2.897586898$
$x_3 = 0.8$	$y_3 = 3.166040571$
$x_4 = 1.0$	$y_4 = 3.559752813$
$x_5 = 1.2$	$y_5 = 4.069834452$
$x_6 = 1.4$	$y_6 = 4.70417746$

Week 4

Topic: Richardson Extrapolation

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RICHARDSON EXTRAPOLATION: FOUR IDEAS

Error Reduction

Uses results from two calculations to estimate and reduce error in numerical solutions.

Predictive Correction

Predicts the behavior of error terms to improve estimates.

Higher Accuracy

Increases accuracy by combining approximations with different step sizes.

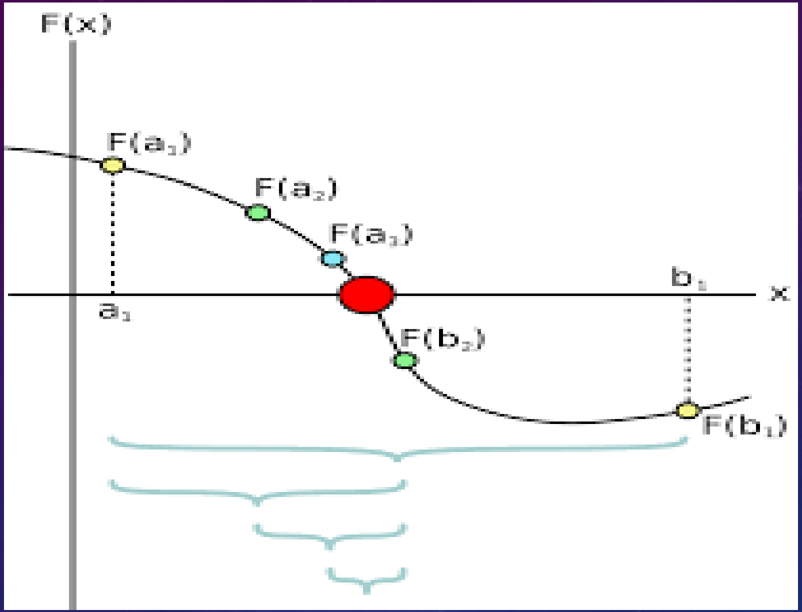
Iterative Refinement

Repeatedly refines results for better precision in numerical computations.

Week: 5

Topic: Bisection Method

Page: 12-13



Question: Determine the root of $x^2 - 3 = 0$

for $x \in [1,2]$by using **Bisection Method**.

Solution: Given: $x^2 - 3 = 0$ Let $f(x) = x^2 - 3$

Now, find the value of $f(x)$ between $a = 1$ and$b = 2$.

$f(1) = 1^2 - 3 = -2 < 0$ and $f(2) = 2^2 - 3 = 1 > 0$

The given function is continuous, and the root lies in the interval [1, 2].

The iterations for the given functions are:

Iterations	a	b	$c = \frac{a+b}{2}$	$f(c)$
1	1.0	2.0	1.5	–0.7500
2	1.5	2.0	1.75	0.06250
3	1.5	1.75	1.625	–0.3594
4	1.625	1.75	1.6875	–0.1523
5	1.6875	1.75	1.71875	–0.04590
6	1.71875	1.75	1.734375	0.008057
7	1.71875	1.734375	1.7265625	–0.01898
8	1.7265625	1.734375	1.7304688	–0.005478
9	1.7304688	1.734375	1.7324219	0.001286
10	1.7304688	1.7324219	1.7314453	–0.002097
11	1.7314453	1.7324219	1.7319336	–0.000406
12	1.7319336	1.7324219	1.7321777	0.0004397
13	1.7319336	1.7321777	1.7320557	0.0000168

Since $|x_{13} - x_{12}| = 1.22 \times 10^{-4} < 1 \times 10^{-3}$. Hence, 1.7320557 is the approximated solution.

Question: Find the root of the equation $e^{-x}(3.2\ sin\ x - 0.5\ cos\ x) = 0$ for $x \in [0,0.5]$ by using **Bisection Method.**

Solution: Let $f(x) = e^{-x}(3.2\ sin\ x - 0.5\ cos\ x)$.

Now, find the value of $f(x)$

at $a = 0$

and$b = 0.5$.

$f(0) = e^{-0}(3.2\ sin\ 0 - 0.5\ cos\ 0) =$

$-0.5 < 0$

and $f(0.5) = e^{-0.5}(3.2\ sin\ 0\ .\ 5 - 0.5$

$cos\ 4) = 0.664 > 0$

The given function is continuous, and the

root lies in the interval [0, 0.5].

The iterations for the given functions are:

Iterations	a	b	$c = \frac{a+b}{2}$	$f(c)$
1	0	0.5	0.25	0.239276
2	0	0.25	0.125	–0.085725
3	0.125	0.25	0.1875	0.087259
4	0.125	0.1875	0.15625	0.003472
5	0.125	0.15625	0.140625	–0.040440
6	0.140625	0.15625	0.1484375	–0.018314
7	0.1484375	0.15625	0.15234375	–0.007378
8	0.15234375	0.15625	0.15429688	–0.001943
9	0.15429688	0.15625	0.15527344	0.000767
10	0.15429688	0.15527344	0.15478516	–0.000587
11	0.15478516	0.15527344	0.15502930	0.000090
12	0.15478516	0.15502930	0.15490723	–0.000248
13	0.15490723	0.15502930	0.15496826	–0.000079

Since $|x_{13} - x_{12}| = 6.1 \times 10^{-5} < 1 \times 10^{-4}$. Hence, 0.154968 is the approximated solution.

Exercise: Find the values of the following functions by using Bisection method.

Week: 6

Topic: False Position Method

Page: 14

Iterations	a	b	$c = \dfrac{a\ f(b) -}{f(b) -}$	$f(c)$
1	1.20000000	1.80000000	1.53767123	–0.80390181
2	1.53767123	1.80000000	1.59219345	–0.11171146

3	1.59219345	1.80000000	1.59950342	–0.01463191
4	1.59950342	1.80000000	1.60045633	–0.00190145
5	1.60045633	1.80000000	1.60058008	–0.00024684

6	1.60058008	1.80000000	1.60059615	–0.00003204
7	1.60059615	1.80000000	1.60059823	0.0000041588
8	1.60059823	1.80000000	1.60059850	–0.000000539

Since $|x_8 - x_7| = 2.7 \times 10^{-7} < 1 \times 10^{-6}$.

Hence, 0.154968 is the approximated solution.

Exercise: Find the values of the following functions by using False Position method.

$(i) \quad x^3 - 2x - 5 = 0$

$(ii) \quad 3x - \cos x - 1 = 0$

$(iii) \quad x^2 - \ln x - 12 = 0$

Week: 7

Topic: Newton-Raphson Method Page: 15-16

Newton-Raphson Method

Algorithm of Newton-Raphson Method:

1. Define a function $f(x)$
2. Find points a and b such that $a < b$ and $f(a) \times f(b) < 0$

3. Take the interval $[a, b]$ and find next value $x_0 = \frac{a+b}{2}$

4. Find the derivative of the function $f'(x)$

5. Find $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

6. If $f(x_n) = 0$ then x_n is an exact root, otherwise $x_n = x_{n+1}$

7. Repeat the step 5 and 6 until $f(x_i) = 0$, $i = 1, 2, 3 \cdots \cdots$

or $|f(x_i)| \leq$ Accuracy.

Question: Find the positive root of $x = \cos x$ for $x \in [0,1]$ using **Newton-Raphson method**.

Let $f(x) = x - \cos x \therefore f'(x) = 1 + \sin x$

Here $f(0) = -1 < 0$ and $f(1) = 0.4596769 > 0$

The given function is continuous, and the root lies in the interval [0, 1].

Here, $x_0 = \frac{0+1}{2} = 0.5$.

$x_4 = 0.739085133$

Since $|x_4 - x_3| = 4 \times 10^{-9} < 1 \times 10^{-8}$.

Hence the required root is 0.739085133 **Home works:**

1. $x^3-2x^2-4=0$, x € [2,3], correct up to 5 decimal places

2. $x^4+x^2-80=0$, x€ [2,3], correct up to 3

3. e^{-x}=10x, x€ [0, 0.2], correct up to 8 decimal places

4. $x = \sqrt[3]{48}$, x€ [3,2.4], correct up to 8 decimal places

Exercise: Find the values of the following functions by using Newton-Raphson method.

$(i) \quad f'(x) = 6x^2 - 3 \quad (ii) f'(x)$

$= x \;\; e^x + e^x + sin\, x \quad (iii) f'(x)$

$= 3 + sin\, x \quad (iv) f'(x)$

$= 10 - cos\, x$

Week: 8

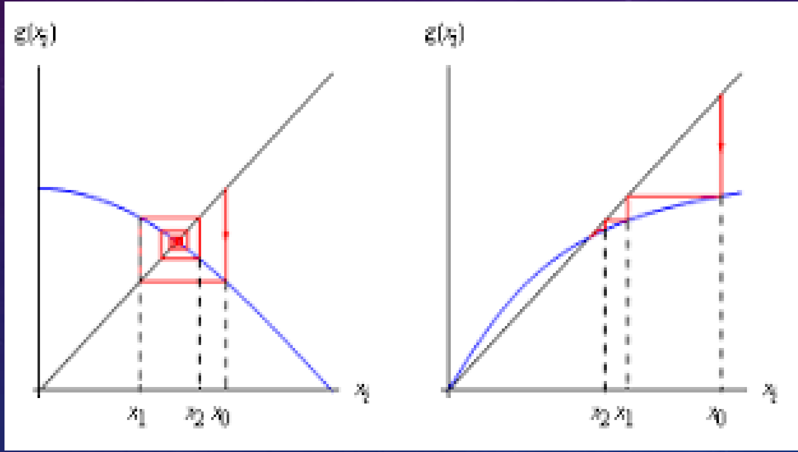
Topic: <u>Fixed Point Iteration</u>

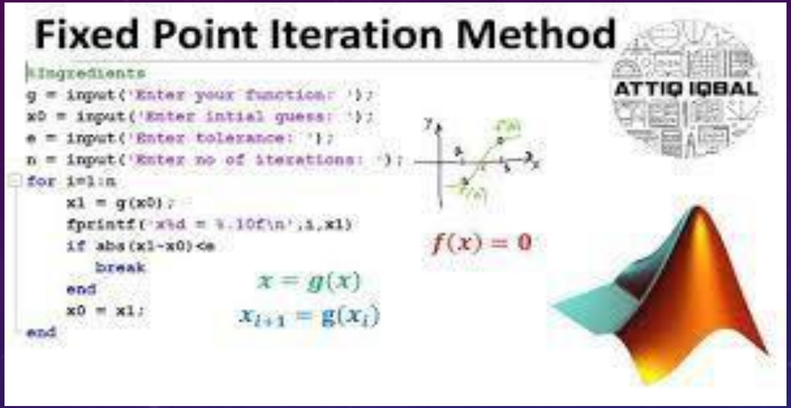
<u>Method</u> Page: 16-17

Algorithm of Fixed Point Iteration Method:

1. Define a function $f(x)$

2. Find points a and b such that $a < b$ and $f(a) \times f(b) < 0$





<u>Question:</u> Find the root of $x^2 - x - 1 = 0$ for $x \in [1,2]$by using **Fixed Point Iteration** method.

Since $|x_8 - x_7| = 2.23 \times 10^{-5} < 1 \times 10^{-4}$.
Hence the desired root is 1.61802401

Home works:

Exercise: Find the values of the following functions by using Fixed Point Iteration method.

Topic: Triangular systems and back substitution, Forward and Backward Substitution

Page: 18-19

A triangular system refers to a system of linear equations where the coefficient matrix is either upper or lower triangular. In an upper triangular system, all the elements below the diagonal are zero, while in a lower triangular system, all elements above the diagonal are zero. Solving triangular systems is efficient because it allows for straightforward methods like forward and backward substitution.

Forward Substitution is used when solving a system with a lower triangular matrix. Here's how it works:

1. Start with the first equation, which involves only one variable.
2. Solve for that variable.
3. Substitute the value into the next equation, which will have two variables.
4. Continue substituting solved values into subsequent equations until all variables are determined.

This method is called "forward" because we solve for variables from top to bottom, progressing through the equations in sequence.

Backward Substitution, on the other hand, is used for upper triangular systems. Here’s the process:

1. Begin with the last equation, which contains only one variable.
2. Solve for this variable.
3. Substitute the value into the second- to-last equation, which will have two variables.
4. Continue substituting solved values into previous equations until all variables are determined.

Backward substitution is called so because we solve for variables starting from the bottom and working upwards, utilizing already-known values for higher variables.

Both substitution methods are efficient and provide a systematic approach to solving triangular systems. They are commonly used in numerical methods and algorithms like Gaussian elimination, especially when dealing with large systems of equations where direct methods may be computationally expensive.

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In summary, triangular systems and the associated forward and backward substitution methods simplify the solution of linear systems by breaking down the equations in a step-by-step manner.

Week: 10

Topic: Gauss elimination method Page: 20

1. Solve Equations 2x+5y=21,x+2y=8 using Gauss-Jordan Elimination method

Solution:
Total Equations are 2

$2x + 5y = 21 \longrightarrow (1)$

$x + 2y = 8 \longrightarrow (2)$

Converting given equations into matrix form

$$\left[\begin{array}{cc|c} 2 & 5 & 21 \\ 1 & 2 & 8 \end{array}\right]$$

$R_1 \leftarrow R_1 \div 2$

$$= \left[\begin{array}{cc|c} 1 & \frac{5}{2} & \frac{21}{2} \\ 1 & 2 & 8 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$$= \left[\begin{array}{cc|c} 1 & \frac{5}{2} & \frac{21}{2} \\ 0 & -\frac{1}{2} & -\frac{5}{2} \end{array}\right]$$

$R_2 \leftarrow R_2 \times -2$

$$= \left[\begin{array}{cc|c} 1 & \frac{5}{2} & \frac{21}{2} \\ 0 & 1 & 5 \end{array}\right]$$

$R_1 \leftarrow R_1 - \frac{5}{2} \times R_2$

$$= \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

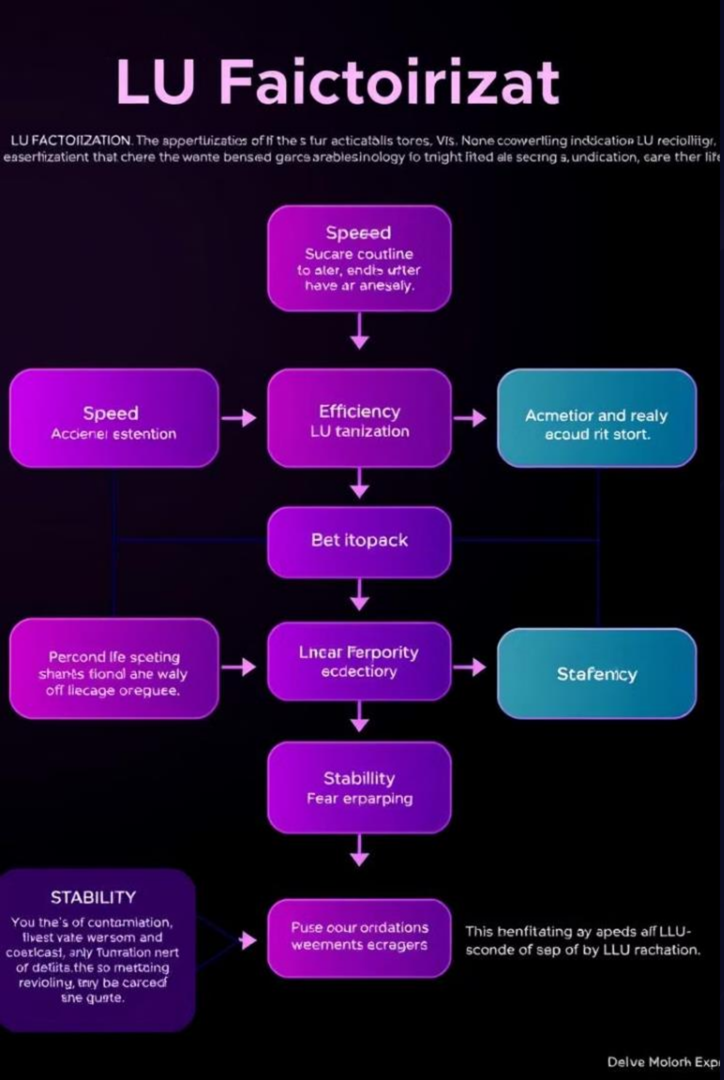
Week: 11

Topic: LU Factorization Method Page: 21-26

Question: Solve the following system of linear equations using LU Factorization Method.

$x + 2y + 4z = 3, 3x + 8y + 14z = 13$

$2x + 6y + 13z = 4$



Advantages of LU Factorization

1 Faster Solving

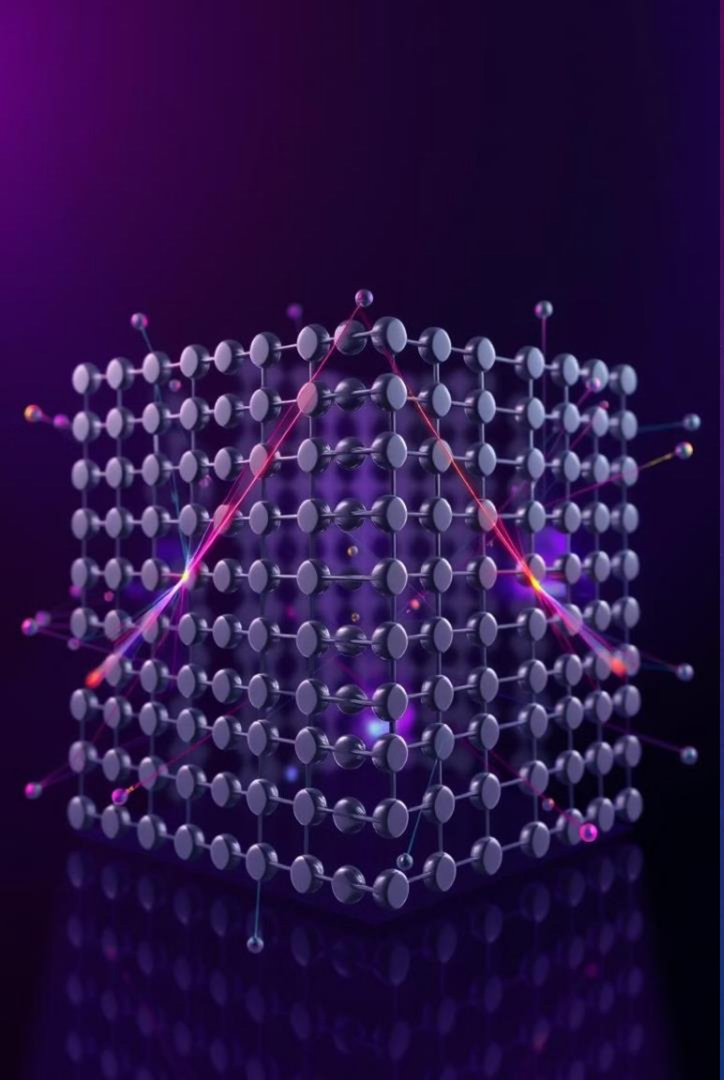
LU Factorization provides a quicker way to solve linear equations compared to Gaussian elimination, especially for larger systems.

2 Improved Stability

LU Factorization is less prone to numerical instability, making it more reliable for solving systems with potential round-off errors.

3 Multiple Solutions

Once you decompose a matrix into LU, you can efficiently solve for multiple right-hand sides (b) with the same coefficient matrix A, saving computational time.



LU Factorization: A Powerful Matrix Decomposition

LU Factorization is a fundamental technique in linear algebra that breaks down a matrix into a lower triangular matrix (L) and an upper triangular matrix (U). This decomposition allows us to solve linear systems more efficiently and perform other matrix operations with greater ease.

What is LU Factorization?

Definition

LU Factorization represents a square matrix A as the product of a lower triangular matrix L and an upper triangular matrix U, where L is unit lower triangular (diagonal elements are all 1).

Equation

$A = LU$

Calculating LU Factorization

Step 1

Begin with the original matrix A.

Step 2

Perform Gaussian elimination on A to obtain an upper triangular matrix U, keeping track of the multipliers used.

Step 3

Construct L using the multipliers from the elimination process, ensuring its diagonal elements are 1.

Example 1: LU Factorization of a 3x3 Matrix

1

Matrix A

The original matrix.

2

LU Decomposition

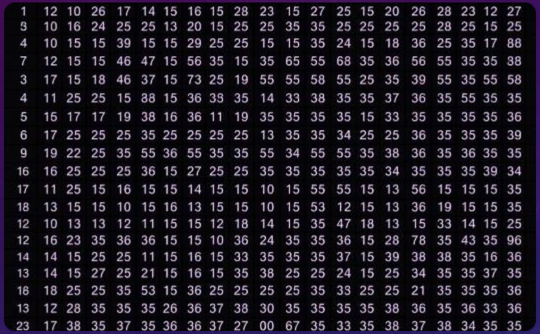
Performing Gaussian elimination and constructing L and U.

3

Verification

Confirming A = LU by multiplying L and U.

Visualizing LU Factorization



Matrix A

A representation of the original matrix.

Lower Triangular Matrix L

Represents the elimination operations.

Upper Triangular Matrix U

The resulting upper triangular matrix after elimination.

Applications of LU Factorization

Solving Linear Systems

LU Factorization is a cornerstone of solving linear systems of equations, particularly for large-scale applications.

Finding Inverses

LU Factorization can be used to find the inverse of a matrix, a fundamental operation in many matrix-based problems.

Calculating Determinants

The determinant of a matrix can be efficiently calculated using the LU factorization.

Limitations and Considerations

1

Non-Invertible Matrices

LU Factorization cannot be performed on non-invertible matrices.

2

Numerical Instability

While LU Factorization is generally stable, certain matrices can lead to numerical instability.

3

Computational Efficiency

While efficient for solving systems, LU Factorization can be computationally expensive for extremely large matrices.

LU Factorization vs. Other Matrix Decompositions

LU Factorization

Efficient for solving linear systems.

QR Factorization

Used for least-squares problems and eigenvalue problems.

Cholesky Decomposition

Applies only to symmetric positive definite matrices.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Pros	• Lur prcotle ffactiashem fraceralllor • Lor factorsatal facct fin randon	• Lur coractoy. Ifranchem decortzation. • Lur fauceccsest fact fin ecrouter.
	• Lecceest di anostions	• Incoucce llanicn loceagee ool rtitions 15t raution.
Pons	• Lescnulcesfficting • Ng spofer audheorisfact randon	• Leromslforning • Nipl cosses and lepsion raution.
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Week: 12

Topic:

Pivoting Page:

27

Pivoting (Basic Concept)

Pivoting is a technique used in numerical methods, particularly in solving systems of linear equations. It involves selecting a **pivot element** in the matrix and using it to simplify the system of equations. The pivoting process helps to improve the stability of the algorithm and avoid errors, such as dividing by very small numbers or zero.

There are two main types of pivoting:

1. **Partial Pivoting**: This involves selecting the largest element in the

current column (in absolute value) to be the pivot. This helps reduce numerical errors by ensuring that the pivot is large, thus avoiding small numbers in the calculations.

2. **Complete Pivoting**: This is a more advanced form, where the largest element in the entire submatrix (both in the current row and column) is selected as the pivot. This provides even more stability, but it requires more computational work since it searches both rows and columns for the largest value.

In summary, **pivoting** ensures that the algorithm uses the most suitable elements in the matrix to prevent numerical issues and get accurate results when solving equations.

Week: 13+14

Topic: Forward and backward difference

Page: 27-32

Newton-Gregory Forward Difference formula

Gregory Newton's is a forward difference formula which is applied to calculate finite difference identity

$$(x-a)(x-b)(x-c)$$
$$= x^3 - (a+b+c)x^2$$
$$+ (ab+bc+ca)x - abc$$

Question: The table below gives the values of $tan\ x$ for $0.10 \leq x \leq 0.30$.
Estimate the value of (a) $tan(0.12)$ (b) $tan(0.26)$ (c) $tan(0.40)$

x	0.10	0.15	0.20	0.25	0.30
$y = tan\ x$	0.1003	0.1511	0.2027	0.2553	0.3093

Solution: We construct the difference table for the given data is as follows:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.10	**0.1003**	**0.0508**	**0.0008**	**0.0002**	**0.0002**
0.15	0.1511	0.0516	0.0010	0.0004	

0.20	0.2027	0.0526	0.0014		
0.25	0.2553	0.0540			
0.30	0.3093				

Year x	1891	1901	1911	1921	1931
Population(in thousand) y	46	66	81	93	101

Solution: We construct the difference table for the given data is as follows:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$

1891	**46**	**20**	**-5**	**2**	**-3**
1901	66	15	-3	-1	
1911	81	12	-4		
1921	93	8			
1931	101				

By Newton Forward Difference formula, we have

$$y(1895) = 46 + 0.4 \times 20 + \frac{0.4(0.4-1)}{2!} \times (-5) + \frac{0.4(0.4-1)(0.4-2)}{3!} \times 2 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} \times (-3) = 54.853$$

The population for the year 1895 is 54.853 thousands approximately.

Exercise-1: From the following table of premium for policies maturing at different ages, estimate the premium for policies maturing at age 46 and 63.

Age x	45	50	55	60	65
Premium y	114.84	96.16	83.32	74.48	68.48

Exercise-2: Estimate using forward difference method the increase in the population during the year from 1946 to 1948.

Year	19 11	19 21	19 31	19 41	19 51	19 61
Population(thou sands)	12	15	20	27	39	52

Question-1: Marks obtain by students in an examination are given below. Estimate the number of students who obtained less than
(a) 42 marks (b) 90 marks.

Marks obtain (less than)	20	40	60	80	100
No. of students	41	103	168	218	235

Solution: We construct a backward difference table of the given data is as follows:

Marks obtain (less than) x	No. of students $f(x)$	$\nabla f(x$	$\nabla^2 f(x$	$\nabla^3 f(x$	$\nabla^4 f(x$
20	41	62	3		
40	103	65			
60	168				

$$f(42) = 41 + 1.1 \times 62 + \frac{1.1(1.1-1)}{2!} \times 3 + \frac{1.1(1.1-1)(1.1-2)}{3!} \times (-18)$$

$$= 110$$

Hence the number of students who get less than 42 marks is 110.

(b) From the table we shall find $f(90)$, the number of students who get less than 90 marks.

In this case, we use backward difference formula. $f(x) = f(x_n) + u\ \nabla f(x_n) +$

Question-2: The following data satisfying $y = f(x)$. Find the Backward difference formula.

x	1	2	3	4	5	6
$f(x)$	0	15	48	105	192	315

Solution: We construct a backward difference table of the given data is as follows:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	0				

2	15	15	18	6	0
3	48	33			
4	105	57	24		
5	192	87	30	6	
6	**315**	**123**	**36**	**6**	**0**

By Newton Backward Difference formula, we have

$= 315 + 123x - 738 + 18x2$
$- 198x + 540 + x^3 - 4x^2 - 11x^2$
$+ 44x + 30x - 120$
$= x^3 + 3x^2 - x - 3$

Question-3: The following data satisfying $y = f(x)$. Find the Backward difference formula.

x	1	2	3	4	5	6
$f(x)$	2	6	20	50	102	182

Solution: We construct a backward difference table of the given data is as follows:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	2	4	10	6	
2	6				
3	20	14			
4	50	30	16		

x	1941	1951	1961	1971	1981	1991
y	2500	2800	3200	3700	4350	5225

Exercise-2: Marks obtain by students in an examination are given below. Estimate the number of students who obtained less than
(a) 45 marks (b) 78 marks.

Marks obtain (less than)	40	50	60	70	80

No. of students	31	73	124	159	190

Week: 15

Topic: Divided difference Page: 33-35

Question-1: Given the following data satisfying $y = f(x)$. Find the Divided difference formula.

x	-1	0	3	6	7
y	3	-6	39	822	1611

Solution: Since the interval is unequal, so we construct the following divided difference table:

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0)$$
$$+ (x - x_0)(x - x_1)\Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x_0)$$
$$+(x - x_0)(x - x_1)(x - x_2)(x$$
$$- x_3)\Delta^4 f(x_0)$$

$$= 3 + (x + 1)(-9) + (x + 1)(x - 0)$$
$$\cdot 6 + (x + 1)(x - 0)(x - 3) \cdot 5$$
$$+ (x + 1)(x - 0)(x - 3)(x - 6) \cdot 1$$

$$= 3 - 9x - 9 + 6(x^2 + x)$$
$$+ 5x(x^2 - 2x - 3)$$
$$+ (x^2 + x)(x^2 - 9x + 18)$$

$$= 3 - 9x - 9 + 6x^2 + 6x + 5x^3$$
$$- 10x^2 - 15x + x^4 - 9x^3 + 18x^2$$
$$+ x^3 - 9x^2 + 18x$$
$$= x^4 - 3x^3 + 5x^2 - 6$$

Question-2: Given the following data satisfying$y = f(x)$. Find the Divided difference formula.

x	0	2	3	4	7	9
y	4	26	58	112	466	922

Solution: Since the interval is unequal, so we construct the following divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$

0	4	$\frac{26-4}{2-0} = 11$	$\frac{32-11}{3-0} = 7$	$\frac{11-7}{4-0} = 1$	$\frac{1-1}{7-1} = 0$
2	26				
3	58	$\frac{58-26}{3-2} = 32$	$\frac{54-32}{4-2} = 11$	$\frac{16-1}{7-2} = 1$	$\frac{1-1}{9-2} = 0$
4	112	$\frac{112-58}{4-3} = 54$			

7	446	$\frac{466-12}{7-4} = 118$	$\frac{466-12}{7-3} = 16$	$\frac{22-1}{9-3} = 1$	
9	912	$\frac{922-46}{9-7} = 228$	$\frac{228-11}{9-4} = 22$		

By Newton divided difference formula, we have

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 f(x_0)$$

$$= 4 + (x - 0) \times 11 + (x - 0)(x - 2) \times 7 + (x - 0)(x - 2)(x - 3) \times 1 + (x - 0)(x - 2)(x - 3)(x - 4) \times 0$$

$= 4 + 11x + 7(x^2 - 2x)$
$+ x(x^2 - 5x + 6)$
$= 4 + 11x + 7x^2 - 14x + x^3 -$
$5x^2$
$+ 6x$
$= x^3 + 2x^2 + 3x + 4$

<u>Question-3:</u> Given the following data satisfying $y = f(x)$. Find the Divided difference formula.

x	1	3	4	6	7
y	-3	9	30	132	225

Solution: Since the interval is unequal, so we construct the following divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	-3	$\frac{9+3}{3-1} = 6$	$\frac{21-6}{4-1} = 5$	$\frac{10-5}{6-1} = 1$	
3	9	$\frac{30-9}{4-3} = 21$			

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0)$$
$$+ (x - x_0)(x - x_1)\Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x_0)$$
$$+(x - x_0)(x - x_1)(x - x_2)(x$$
$$- x_3)\Delta^4 f(x_0)$$
$$= -3 + (x - 1) \times 6$$
$$+ (x - 1)(x - 3) \times 5$$
$$+ (x - 1)(x - 3)(x - 4) \times 1$$
$$+ (x - 1)(x - 3)(x - 4)(x - 6)$$
$$\times 0$$
$$= -3 + 6x - 6 + 5(x^2 - 4x + 3)$$
$$+ (x - 1)(x^2 - 7x + 12) + 0$$

$$= -3 + 6x - 6 + 5x^2 - 20x + 15$$
$$+ x^3 - 7x^2 + 12x - x^2 + 7x - 12$$
$$= x^3 - 3x^2 + 5x - 6$$

Exercise-1: Given the following data satisfying $y = f(x)$. Find the Divided difference formula.

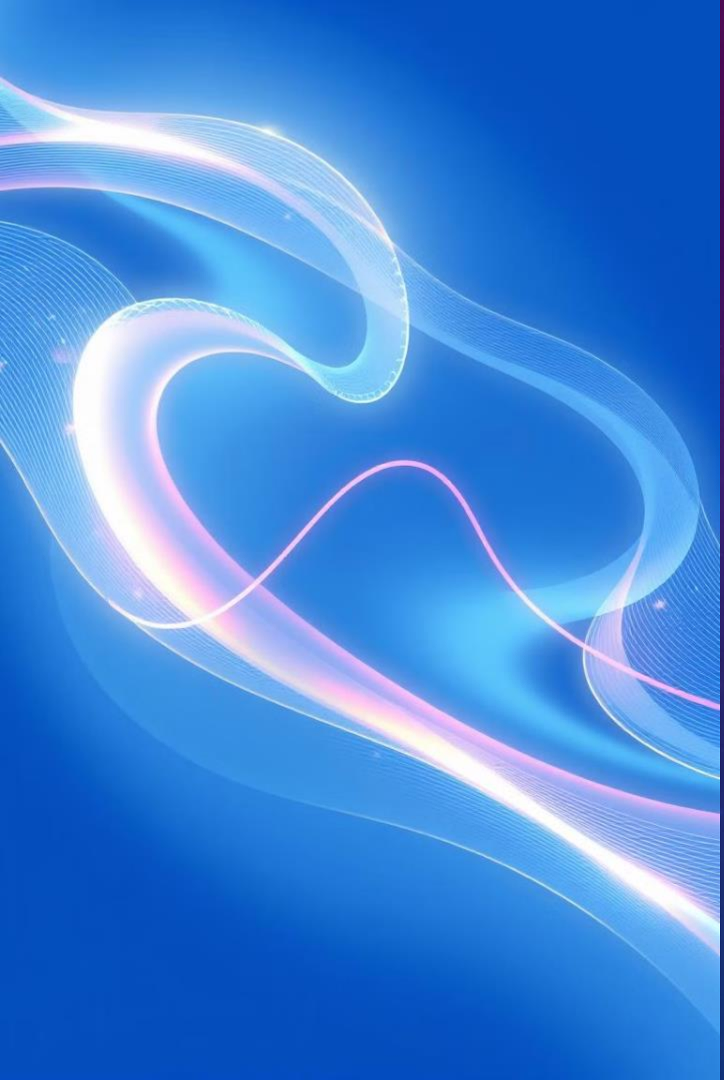
x	4	5	7	10	11	13
y	48	100	294	900	1210	2028

Exercise-2: Given the following data satisfying $y = f(x)$. Find the Divided difference formula.

Week: 16+17

Topic: Lagrange

Polynomial Page: 36-38



Lagrange Polynomial: A Fascinating Mathematical Odyssey

Prepare to embark on a journey into the intricate world of Lagrange polynomial, a powerful mathematical tool that unlocks a wealth of possibilities.

Unraveling the Essence of Lagrange Polynomial

Interpolating Data

Imagine a set of data points plotted on a graph. Lagrange polynomial lets you create a smooth curve that passes precisely through these points, even when the data is scattered.

Unique Solution

For any given set of data points, there's only one Lagrange polynomial that can perfectly fit them. This uniqueness makes it a reliable tool for analysis and prediction.

Practical Applications and Real-World Examples

Engineering

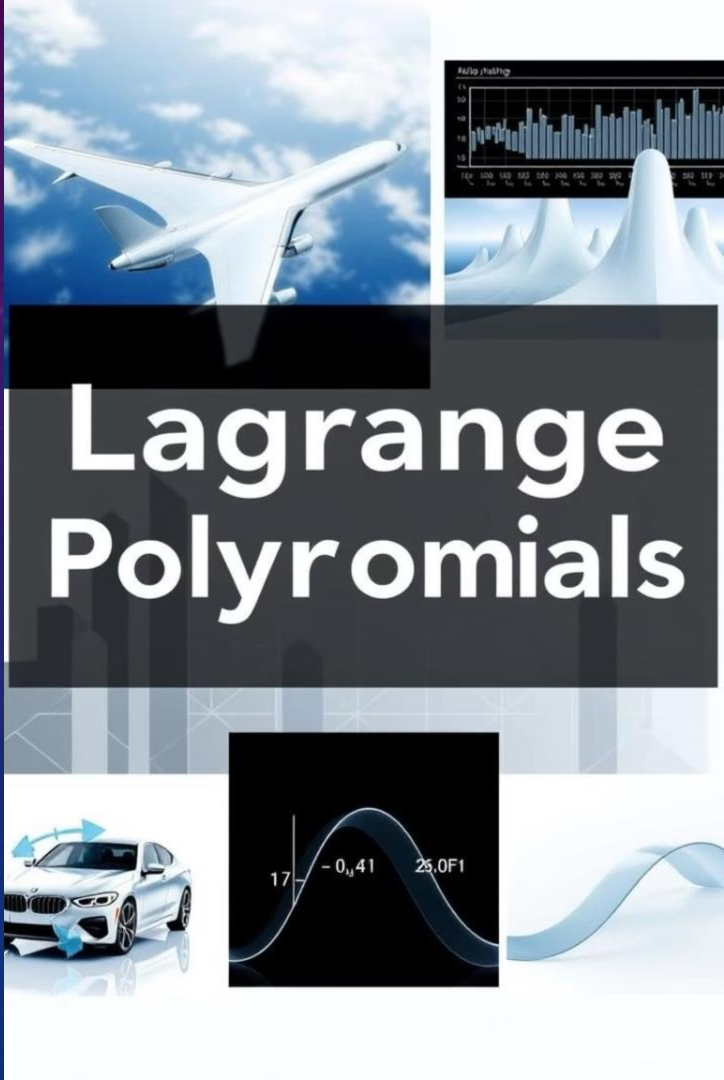
Used for designing complex shapes in aerospace engineering, optimizing robot trajectories, and analyzing data in mechanical simulations.

Finance

Assists in modeling stock prices, forecasting market trends, and creating financial models for risk management and portfolio optimization.

Computer Graphics

Plays a crucial role in creating realistic animations and special effects in movies, games, and virtual reality experiences.



The Beauty of Lagrange Polynomial Unveiled

1

Elegant Formula

The equation itself is a masterpiece of mathematical elegance, showcasing the power of simple yet profound concepts.

2

Versatile Application

From scientific research to everyday technology, Lagrange polynomial finds its place in countless fields, proving its versatility.

3

Unveiling Insights

It allows us to gain deeper insights into complex phenomena and make accurate predictions about the future.

Exercise-1: Given the following data satisfying $y = f(x)$. Find the Lagrange Polynomial.

x	1	3	4	6
y	-10	36	110	420

Exercise-2: Given the following data satisfying $y = f(x)$. Find the Lagrange Polynomial.

x	1	3	5	6
y	18	-8	30	88

Exercise-3: Given the following data satisfying $y = f(x)$. Find the Lagrange Polynomial.

x	1	3	4	6
y	-3	9	30	132

Answer: 1. $f(x) = 2x^3 + x^2 - 7x - 6$

2. $f(x) = x^3 - x^2 - 22x + 40$

3. $f(x) = x^3 - 3x^2 + 5x - 6$

Thank You!