University of Global Village (UGV)

Course Title: Linear Algebra and Complex Analysis
Course Code: 0541-2207
Credit: 3
CIE: 90 marks
SEE: 60 Marks
Exam hour: 3

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Course Outlines

Course Code: Math 0541-2207 Semester End Exam (SEE): 3 Hours	Credit: 03 CIE Marks: 90 SEE marks: 60
Course Learning Outcomes (CLO): After successful completion of the course students will be able to -	
CLO1 Describe Matrices, Type of matrices, various operations of matrices, System of linear Equations and Different applications.	
CLO2 Apply the acquired concepts of Linear algebra and its applications in engineering.	
CLO3 Define the basic terminology and theorems associated with Linear algebra and Complex Variables.	
CLO4 Properties of Complex Numbers, Complex Functions, Related Mathematics and various theorems and it's applications.	

Course Content Summary

SL.	Content of Courses	Hrs	CLO's
1	Matrix, its Operations and classifications, Symmetric and skew-Symmetric matrix and related theorems, Orthogonal, Involutary and Idempotent matrices and related mathematics, Determinant and Inverse Matrix.	8	CLO1, CLO3
2	System of Linear Equations, Matrix method for solving linear system, Cramers rule for solving linear systems and solving related problems, Rank of a matrix and eigenvalue, Cayley-Hamilton theorem, Application of Linear Algebra in Engineering and related mathematics.	10	CLO2, CLO3
3	Definition of complex number, Modulus and argument of complex number, Polar Form of Complex Number, Related Mathematics, Graphical Representation of Complex Number, Circle and Ellipse related Problems, Analytic Function, Entire Function, Singularity, different types of singularity.	8	CLO3, CLO4
4	Definition of Complex Integral, Cauchy Integral Formula for Derivatives, Finding Integration using CIF, Related Mathematics, Harmonic Function, Finding harmonic Conjugate, Laurent's Theorem, Residue, Cauchy Residue Theorem, CRT Related mathematics.	8	CLO3, CLO4

Course Plan Specifying content, CLO's, Teaching Learning, and Assessment strategy mapping with CLO's

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
1	Matrix • Definition • its operations and • classifications.	Lecture, Discussion	Quiz	CLO1
2	Symmetric and skew-Symmetric matrix • Definition • related theorems • related mathematical problems	Discussion, Oral Presentation	Written Assignment	CLO1
3	Orthogonal, Involutary and Idempotent matrices • Definition • related mathematics	Oral Presentation	Oral Presentation	CLO1
4	Determinant and Inverse Matrix • Definition • Methodology • related mathematics	Group Work	Group Assignment	CLO1
5	System of Linear Equations • Definition • Methodology • related mathematics	Case Study	Presentation	CLO2

	Matrix method for solving linear systems. • Methodology • related mathematics			
6	Cramer's rule • Methodology • related mathematics	Group Work	Quiz, Written Assignment	CLO2
7	Rank of a matrix and eigenvalue • Definition • related theorems • related mathematical problems	Lecture, Discussion	Oral Presentation, Quiz	CLO1
8	Eigenvalue and eigenvector Cayley-Hamilton theorem • theorem • verification • mathematical problems	Discussion, Oral Presentation	Group Assignment, Quiz	CLO3
9	Application of Linear Algebra in Engineering and related mathematics.	Oral Presentation	Presentation, Written Assignment	CLO2
10	Complex Number • Definition of complex number • Modulus and argument of complex number	Oral Presentation	Quiz, Presentation	CLO3

	• Polar Form of Complex Number • Related Mathematics.			
11	Graphical Representation of Complex Number • Circle • Ellipse • related Problems.	Group Work	Written Assignment, Oral Presentation	CLO3
12	• Analytic Function • Entire Function • Singularity • different types of singularity	Discussion, Oral Presentation	Group Assignment, Presentation	CLO3
13	• Definition of Complex Integral • Cauchy Integral Formula for Derivatives • Finding Integration using CIF • Related Mathematics.	Discussion, Oral Presentation	Quiz, Group Assignment	CLO4
14	• Harmonic Function • Finding harmonic Conjugate	Oral Presentation	Written Assignment, Quiz	CLO4
15	Harmonic Function Related Mathematics.	Lecture, Discussion	Oral Presentation, Group Assignment	CLO4
16	• Laurent's Theorem • Residue	Practical Work	Presentation, Quiz	CLO4

	• Cauchy Residue Theorem			
17	CRT Related mathematics.	Reading Assignment	Quiz, Written Assignment, Oral Presentation	CLO4

REFERENCE BOOKS

1. Complex Variables– Schaum's Outline series.
2. Complex analysis- Titas Math series.
3. Linear Algebra- Prof. Dr. Abdur Rahman
4. Linear Algebra- Haward Anton.
5. Linear Algebra- Schaum's Outline series

Assessment Pattern

CIE- Continuous Internal Evaluation (90 Marks)

Bloom's Category Marks (out of 60)	Tests (45)	Assignments (15)	Quizzes (15)	Attendance (15)
Remember	05		05	
Understand	05		05	
Apply	10	05	05	15
Analyze	10	05		
Evaluate	10	05		
Create	05			

SEE- Semester End Examination (60 Marks)

Bloom's Category	Test
Remember	10
Understand	10
Apply	10
Analyze	10
Evaluate	15
Create	5

LECTURE-LINEAR ALGEBRA

- Week 1
- Topic: Matrix, its types and operations
- Page no. (6-31)

- Matrix

- A matrix is an array of numbers arranged in the form of rows and columns. The number of rows and columns of a matrix are known as its dimensions which is given by m

- $\times$ n, where m and n represent the number of rows and columns respectively. Apart from basic mathematical operations, there are certain elementary operations that can be performed on a matrix. The elementary operations or transformation of a matrix are the operations performed on rows and columns of a matrix to transform the given matrix into a different form to make the calculation simpler. In this article, we are going to learn three basic elementary operations of matrix in detail with examples.

- Three Basic Elementary Operations of Matrix

- We know that elementary row operations are the operations that are performed on rows of a matrix. Similarly, elementary column operations are the operations that are performed on columns of a matrix.

- The three basic elementary operations or transformation of a matrix are:

1. Interchange of any two rows or two columns.
2. Multiplication of row or column by a non-zero number.
3. Multiplication of row or column by a non-zero number and add the result to the other row or column.

- Now, let us discuss these three basic elementary operations of a matrix in detail.

- Case 1: Interchange of any Two Rows or Two Columns

Matrices

Operations of matrices

Types of matrices

Properties of matrices

MATRICES

$$A=\begin{bmatrix}2 & 3 & 7\\ 1 & -1 & 5\end{bmatrix}$$

$$B=\begin{bmatrix}1 & 3 & 1\\ 2 & 1 & 4\\ 4 & & 6\end{bmatrix}$$

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Both *A* and *B* are examples of matrix. A matrix is a rectangular array of numbers enclosed by a pair of bracket.

Why matrix?

MATRICES

Consider the following set of equations:

$$\begin{cases} x+y=7, \\ 3x-y=5. \end{cases}$$

It is easy to show that $x = 3$ and $y = 4$.

How about solving

$$\begin{cases} x+y-2z=7, \\ 2x-y-4z=2, \\ -5x+4y+10z=1, \\ 3x-y-6z=5. \end{cases}$$

Matrices can help…

MATRICES

In the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

- numbers a_{ij} are called *elements*. First subscript indicates the row; second subscript indicates the column. The matrix consists of mn elements
- It is called "the $m \times n$ matrix $A = [a_{ij}]$" or simply "the matrix A " if number of rows and columns are understood.

Square matrices

- When $m = n$, i.e., $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$
- A is called a "square matrix of order n" or "n-square matrix"
- elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ called diagonal elements.
- $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$ is called the *trace* of A.

Equal matrices

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal ($A = B$) iff each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$.
- *iff* pronouns "if and only if"

  if $A = B$, it implies $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$;

  if $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$, it implies $A = B$.

MATRICES

Equal matrices

Example: $A=\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given that $A = B$, find a, b, c and d.

if $A = B$, then $a = 1$, $b = 0$, $c = -4$ and $d = 2$.

MATRICES

Zero matrices

- Every element of a matrix is zero, it is called a zero matrix, i.e.,

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

OPERATIONS OF MATRICES

Sums of matrices

- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then $A + B$ is defined as a matrix $C = A + B$, where $C = [c_{ij}]$, $c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m, 1 \le j \le n$.

Example: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

Evaluate $A + B$ and $A - B$.

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

OPERATIONS OF MATRICES

Sums of matrices

- Two matrices of the same order are said to be *conformable* for addition or subtraction.
- Two matrices of different orders cannot be added or subtracted, e.g.,

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

are NOT conformable for addition or subtraction.

OPERATIONS OF MATRICES

Scalar multiplication

- Let λ be any scalar and $A = [a_{ij}]$ is an $m \times n$ matrix. Then $\lambda A = [\lambda a_{ij}]$ for $1 \le i \le m, 1 \le j \le n$, i.e., each element in A is multiplied by λ.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$. Evaluate $3A$.

$$3A = \begin{bmatrix} 3\times1 & 3\times2 & 3\times3 \\ 3\times0 & 3\times1 & 3\times4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 12 \end{bmatrix}$$

- In particular, $\lambda = -1$, i.e., $-A = [-a_{ij}]$. It's called the *negative* of A. Note: $A - A = 0$ is a zero matrix

OPERATIONS OF MATRICES

Properties

Matrices A, B and C are conformable,

- $A + B = B + A$ (commutative law)
- $A + (B + C) = (A + B) + C$ (associative law)
- $\lambda(A + B) = \lambda A + \lambda B$, where λ is a scalar (distributive law

OPERATIONS OF MATRICES

Properties

Example: Prove $\lambda(A + B) = \lambda A + \lambda B$. Let $C = A + B$, so $c_{ij} = a_{ij} + b_{ij}$.

Consider $\lambda c_{ij} = \lambda\,(a_{ij} + b_{ij}) = \lambda a_{ij} + \lambda b_{ij}$, we have,

$\lambda C = \lambda A + \lambda B$.

Since $\lambda C = \lambda(A + B)$, so $\lambda(A + B) = \lambda A + \lambda B$

OPERATIONS OF MATRICES

Matrix multiplication

- If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then AB is defined as a $m \times n$

matrix $C = AB$, where $C = [c_{ij}]$ with

$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$ for $1 \le i \le m$, $1 \le j \le n$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = AB$. Evaluate c_{21}.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \qquad c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22$$

Matrix multiplication

Example: $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1\times(-1)+2\times2+3\times5=18\\ c_{12} = 1\times2+2\times3+3\times0=8\\ c_{21} = 0\times(-1)+1\times2+4\times5=22\\ c_{22} = 0\times2+1\times3+4\times0=3 \end{cases}$$

$$C = AB = \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix} = \begin{bmatrix}18 & 8\\ 22 & 3\end{bmatrix}$$

Matrix multiplication

- In particular, A is a $1 \times m$ matrix and B is a $m \times 1$ matrix, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

then $C = AB$ is a scalar. $C = \sum_{k=1}^{m} a_{1k} b_{k1} = a_{11} b_{11} + a_{12} b_{21} + \dots + a_{1m} b_{m1}$

OPERATIONS OF MATRICES

Matrix multiplication

- BUT BA is a $m \times m$ matrix!

$$BA = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & b_{11}a_{12} & \dots & b_{11}a_{1m} \\ b_{21}a_{11} & b_{21}a_{12} & & b_{21}a_{1m} \\ \vdots & & \ddots & \\ b_{m1}a_{11} & b_{m1}a_{12} & & b_{m1}a_{1m} \end{bmatrix}$$

- So $AB \neq BA$ in general !

OPERATIONS OF MATRICES

Properties

Matrices A, B and C are conformable,

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $A(BC) = (AB)\, C$
- $AB \neq BA$ in general

However

- $AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$
- $AB = AC$ NOT necessarily imply $B = C$

OPERATIONS OF MATRICES

Properties

Example: Prove $A(B + C) = AB + AC$ where A, B and C are n-square matrices

Let $X = B + C$, so $x_{ij} = b_{ij} + c_{ij}$. Let $Y = AX$, then

$$y_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

So $Y = AB + AC$; therefore, $A(B + C) = AB + AC$

TYPES OF MATRICES

- Identity matrix
- The inverse of a matrix
- The transpose of a matrix
- Symmetric matrix
- Orthogonal matrix

Identity matrix

- A square matrix whose elements $a_{ij} = 0$, for $i > j$ is called upper triangular, i.e., $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$

- A square matrix whose elements $a_{ij} = 0$, for $i < j$ is called lower triangular, i.e., $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$

Identity matrix

- Both upper and lower triangular, i.e., $a_{ij} = 0$, for $i \neq j$, i.e.,

$$D = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

is called a diagonal matrix, simply

$D = \mathrm{diag}[a_{11}, a_{22}, ..., a_{nn}]$

IDENTITY MATRIX

- In particular, $a_{11} = a_{22} = \ldots = a_{nn} = 1$, the matrix is called identity matrix.
- Properties: $AI = IA = A$

Examples of identity matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Special square matrix

- $AB \neq BA$ in general. However, if two square matrices A and B such that $AB = BA$, then A and B are said to be *commute*.

Can you suggest two matrices that must commute with a square matrix A?

Ans: A itself, the identity matrix, ..

- If A and B such that $AB = -BA$, then A and B are said to be *anti-commute*.

Any 2 columns (or rows) of a matrix can be exchanged. If the i^{th} and j^{th} rows are exchanged, it is shown by $R_i \leftrightarrow R_j$ and if the i^{th} and j^{th} columns are exchanged, it is shown by $C_i \leftrightarrow C_j$.

Case 2: Multiplication of Row or Column by a Non-zero Number

The elements of any row (or column) of a matrix can be multiplied by a non-zero number. So if we multiply the i^{th} row of a matrix by a non-zero number *k*, symbolically it can be denoted by $R_i \rightarrow kR_i$. Similarly, for column it is given by $C_i \rightarrow kC_i$.

Case 3: Multiplication of Row or Column by a Non-zero Number and Add the Result to the Other Row or Column

The elements of any row (or column) can be added with the corresponding elements of another row (or column) which is multiplied by a non-zero number. So if we add the i^{th} row of a matrix to the j^{th} row which is multiplied by a non-zero number *k*, symbolically it can be denoted by $R_i \rightarrow R_i + kR_j$. Similarly, for column it is given by $C_i \rightarrow C_i + kC_j$.

Types of Matrices

Various types of Matrices are:

1.Null Matrix: The matrix in which all the elements are zero is known as a null matrix or zero matrix. Generally, it is denoted by '0'. Then, if aij = 0 for all the elements of i and j.

2.Triangular Matrix: The square matrix in which elements above or below the principal diagonal are triangular matrix. If the elements above the principal diagonal are zero then it is a lower triangular matrix and if the elements below the principal diagonal are zero it is an upper triangular matrix.

3.Column Matrix: The matrix which only has one column is known as a column matrix. The order of the column matrix is always seen to be $m \times 1$.

4.Row Matrix: The matrix which only has one row is known as row matrix. The order of the matrix is always seen to be 1Xn.

5.Horizontal Matrix: The matrix with both rows and columns in order of $m \times n$ is a horizontal matrix. In the horizontal matrix number of columns needs to be greater than the number of rows ($n>m$).

6.Vertical Matrix: The matrix with both rows and columns in order $m \times n$ is a vertical matrix. In the vertical matrix Number of rows needs to be greater than the number of columns ($m>n$).

7. Identity Matrix: When all the elements of the principal diagonal are 1 in a matrix then it is said to be an identity matrix or unit matrix.

8. Diagonal Matrix: If all the elements in a square matrix are zero except the principal diagonal is known as a diagonal matrix.

9. Symmetric Matrix: A square matrix which is aij=aji for all values of i and j is known as a symmetric matrix.

Week 2:

Topics- Transpose, Symmetric and skew-symmetric matrix Page no. (32-37)

Transpose matrix:

The new matrix obtained by interchanging the rows and columns of the original matrix is called as the transpose of the matrix. If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A would be the transpose of A. It is denoted by A' or (A^T).

In other words, if $A = \left[a_{ij}\right]_{m\times n}$, then $A = \left[a_{ji}\right]_{n\times m}$.

For example,

If $A = \begin{bmatrix} a & h & d \\ k & b & l \\ g & f & c \end{bmatrix}$. Then $A = \begin{bmatrix} a & k & g \\ h & b & f \\ d & l & c \end{bmatrix}$

Properties:

1) Transpose of Transpose of a Matrix

The transpose of the transpose of a matrix is the matrix itself: $(A^T)^T = A$. Verify that $(A^T)^T = A$.

2) Transpose of a Sum

The transpose of the sum of two matrices is equivalent to the sum of their transposes: $(A + B)^T = A^T + B^T$.

3) Transpose of a Product

The transpose of the product of two matrices is equivalent to the product of their transposes in reversed order: $(AB)^T = B^T A^T$. The same is true for the product of multiple matrices: $(ABC)^T = C^T B^T A^T$.

Question: Suppose $\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix}$. Show that $(AB)^T = B^T A^T$

Solution: Here $\mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 4 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 2+12+1 & 4+0-1 & 8+9+3 \\ 2+0+3 & 4+0-3 & 8+0+9 \\ 1+8+5 & 2+0-5 & 4+6+15 \end{pmatrix} = \begin{pmatrix} 15 & 3 & 20 \\ 5 & 1 & 17 \\ 14 & -3 & 25 \end{pmatrix}$$

Therefore, $(\mathbf{AB})^T = \begin{pmatrix} \mathbf{15} & \mathbf{5} & \mathbf{14} \\ \mathbf{3} & \mathbf{1} & \mathbf{-3} \\ \mathbf{20} & \mathbf{17} & \mathbf{25} \end{pmatrix}$

$$\mathbf{B^T A^T} = \begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{4} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{4} & \mathbf{3} & \mathbf{3} \end{pmatrix} = \begin{pmatrix} \mathbf{15} & \mathbf{5} & \mathbf{14} \\ \mathbf{3} & \mathbf{1} & \mathbf{-3} \\ \mathbf{20} & \mathbf{17} & \mathbf{25} \end{pmatrix}$$

So, $(\mathbf{AB})^T = \mathbf{B^T A^T}$ (Showed)

Exercise: If $\mathbf{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{5} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{4} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} \mathbf{1} & \mathbf{4} & \mathbf{4} \\ \mathbf{3} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$

i. Show that $(A - B)^T = A^T - B^T$
ii. Prove that $(AB)^T = B^T A^T$
iii. Show that $(ABC)^T = C^T B^T A^T$
iv. Find the value of $A^2 + 3B - 4C + 2I$

Solution: (iv) Now, $A^2 = A \cdot A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 4+6+1 & 6+0+2 & 2+9+5 \\ 4+0+3 & 6+0+6 & 2+0+15 \\ 2+4+5 & 3+0+10 & 1+6+25 \end{pmatrix} = \begin{pmatrix} 11 & 8 & 16 \\ 7 & 12 & 17 \\ 11 & 13 & 32 \end{pmatrix}$$

$A^2 + 3B - 4C + 2I$

$$= \begin{pmatrix} 11 & 8 & 16 \\ 7 & 12 & 17 \\ 11 & 13 & 32 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 & 4 \\ 4 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 8 & 16 \\ 7 & 12 & 17 \\ 11 & 13 & 32 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 12 \\ 12 & 0 & 9 \\ 3 & -3 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 16 & 16 \\ 12 & 0 & 4 \\ 8 & -4 & 8 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 11+3-4+2 & 8+6-16+0 & 16+12-16+0 \\ 7+12-12+0 & 12+0-0+2 & 17+9-4+0 \\ 11+3-8+0 & 13-3+4+0 & 32+9-8+2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & -2 & 12 \\ 7 & 14 & 22 \\ 6 & 14 & 35 \end{pmatrix}$$

Symmetric Matrix:

A square matrix 'A' is said to be symmetric if $A=A^T$, that is, matrix A is said to be symmetric if the transpose of matrix A is equal to matrix A itself.

Skew-Symmetric Matrix:

A square matrix A is said to be skew-symmetric if $a_{ij} = a_{ji}$ ($A = -A^T$) for all i and j. In other words, we can say that matrix A is said to be skew-symmetric if transpose of matrix A is equal to negative of matrix A. Note that all the main diagonal elements in the skew-symmetric matrix are zero.

Let's take an example of a matrix $A = \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & -6 \\ 5 & 6 & 0 \end{pmatrix}$

Then $A^T = \begin{pmatrix} 0 & -3 & 5 \\ 3 & 0 & 6 \\ -5 & -6 & 0 \end{pmatrix}$ and $-A^T = \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & -6 \\ 5 & 6 & 0 \end{pmatrix}$ = A. Therefore, $A = -A^T$

Theorem-1: For any square matrix A with **real number** entries, $\boldsymbol{A + A^T}$ is a symmetric matrix and $\boldsymbol{A - A^T}$ is a skew-symmetric matrix.

Proof: Let $B = A + A^T$,

Then $B^T = (A + A^T)^T$

$= A^T + (A^T)^T$ [as $(A + B)^T = A^T + B^T$]

$= A^T + A$ [$as (A^T)^T = A$]

$= A + A^T = B$

Therefore, B = $A + A^T$ is a symmetric matrix.

Now let $C = A - A^T$

Then $C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$

Therefore, C $= A - A^T$ is a skew-symmetric matrix.

Theorem- 2: Any Square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Proof: Let A be a square matrix then, we can write

$$A = \frac{2A}{2} = \frac{A + A^T + A - A^T}{2}$$

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) = S + K$$

From the Theorem 1, we know that $\left(A + A^T\right)$ is a symmetric matrix and $\left(A - A^T\right)$ is a skew-symmetric matrix. Since for any matrix A, $(kA)^T = k\,A^T$, it follows that $\frac{1}{2}\left(A + A^T\right)$ is a symmetric matrix and $\frac{1}{2}\left(A - A^T\right)$ is a skew-symmetric matrix.

Question-1: Show that the following matrix can be written as the sum of a symmetric and a skew-symmetric matrix $A = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix}$.

Solution: Given, $A = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix}$ then, $A^T = \begin{pmatrix} 1 & 5 & 3 \\ 4 & 7 & 0 \\ 5 & -3 & 5 \end{pmatrix}$

Now, the symmetric matrix of A is,

$$S = A + A^T = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 5 & 3 \\ 4 & 7 & 0 \\ 5 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 9 & 8 \\ 9 & 14 & -3 \\ 8 & -3 & 10 \end{pmatrix}$$

Again the skew-symmetric matrix of A is,

$$K = A - A^T = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 5 & 3 \\ 4 & 7 & 0 \\ 5 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Now, $S + K = \begin{pmatrix} 2 & 9 & 8 \\ 9 & 14 & -3 \\ 8 & -3 & 10 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 10 \\ 10 & 14 & -6 \\ 6 & 0 & 10 \end{pmatrix}$

Thus $\frac{1}{2}(S + K) = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix} = A$, i.e. $A = \frac{1}{2}(S + K)$

So, A can be written as the sum of symmetric and skew-symmetric matrices.

Question-2: Show that the following matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

$$P = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & 2 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} -2 & 7 & 9 \\ 4 & -4 & 0 \\ -1 & 3 & -5 \end{pmatrix}$$

Week 3:
Topics: Orthogonal, Idempotent and Involutary Matrix Page no. (36-44)

Orthogonal Matrix:

A square matrix with real numbers or values is termed as an orthogonal matrix if its transpose is equal to the inverse matrix of it. In other words, the product of a square orthogonal matrix and its transpose will always give an identity matrix.

Suppose A is the square matrix with real values, of order n × n. Also, let $A^T$is the transpose matrix of A.

Then according to the definition:

If, $A^T = A^{-1}$ condition is satisfied, then

$$A \cdot A^T = A \cdot A^{-1} = I$$

Where 'I' is the identity matrix of the order n × n. A^{-1} is the inverse of matrix A and 'n' denotes the number of rows and columns. Then we will call A as the orthogonal matrix.

Question-3: Show that the following matrix is the example of an orthogonal matrix

$$P = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

Solution: Given, $P = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}$ then, $P^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

Now, $PP^T = \frac{1}{3} \cdot \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

$$= \frac{1}{9}\begin{pmatrix} 1+4+4 & 2-4+2 & -2-2+4 \\ 2-4+2 & 4+4+1 & -4+2+2 \\ -2-2+4 & -4+2+2 & 4+1+4 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since, $\mathrm{P.P^T} = I$, so P is an Orthogonal matrix.

Question-4: Show that the following matrix is the example of an orthogonal matrix.

$$A = \frac{1}{7}\begin{pmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{pmatrix}$$

Involutory **matrix:**

A Matrix A is said to be Involutory if $A^2 = I$ where, **I** is an Identity matrix.

Question: Show that the following matrix is an Involutory **matrix.**

$$A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

Solution: Given, $A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$

Now $A^2 = A.A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix} . \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 25 - 24 + 0 & 40 - 40 + 0 & 0 + 0 + 0 \\ -15 + 15 + 0 & -24 + 25 + 0 & 0 + 0 + 0 \\ -5 + 6 - 1 & -8 + 10 - 2 & 0 + 0 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since, $A^2 = A.A = I$, so A in an Involutory matrix.

Idempotent **Matrix:** **A matrix A** is said to be Idempotent if $A^2 = A$.

Question: Show that the following matrix is an Idempotent **matrix.**

$$B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$$

Solution: Given, $B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

Now $B^2 = B \cdot B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = B$$

Since, $B^2 = B$, so B in an Idempotent matrix.

The inverse of a matrix

- If matrices A and B such that $AB = BA = I$, then B is called the inverse of A (symbol: A^{-1}); and A is called the inverse of B (symbol: B^{-1}).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Show B is the the inverse of matrix A.

Ans: Note that $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Can you show the details?

The transpose of a matrix

- The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (write A^T).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of A is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

Symmetric matrix

- A matrix A such that $A^T = A$ is called symmetric, i.e., $a_{ji} = a_{ij}$ for all i and j.
- $A + A^T$ must be symmetric. Why?

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

- A matrix A such that $A^T = -A$ is called skew-symmetric, i.e., $a_{ji} = -a_{ij}$ for all i and j.
- $A - A^T$ must be skew-symmetric. Why?

Orthogonal matrix

- A matrix A is called orthogonal if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$

Example: prove that $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal.

Since, $A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$. Hence, $AA^T = A^TA = I$.

Can you show the details?

We'll see that orthogonal matrix represents a rotation in fact!

PROPERTIES OF MATRIX

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^T = A$ and $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

PROPERTIES OF MATRIX

- Example: Prove $(AB)^{-1} = B^{-1}A^{-1}$.
- Since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$ and
- $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = I$.
- Therefore, $B^{-1}A^{-1}$ is the inverse of matrix AB.

Week 4
Topics: Determinant, Inverse
Page no (46-59)

Determinant:
Determinant in linear algebra is a useful value to provide the value of a square matrix. We denote the determinant of any matrix A by det(A), or |A|.

Determinant Formula:

1. Let us take a matrix of order **1** ×**1** as: ***A***= [**2**]
   Then its determinant will be: det(A)=|A| =2

2. Let us take a matrix of 2×2 order as:

$$A = \begin{bmatrix} p & q \\ c & d \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ -2 & 3 \end{bmatrix} = -12 + 10 = -2$$

3. If the matrix is of 3×3 order :

$$B = \begin{pmatrix} p & q & r \\ a & b & c \\ x & y & z \end{pmatrix}$$

Then its determinant will be: Det(B)=|B|= $p(bz - cy) - q(az - cx) + r(ay - bx)$

<u>**Question-5:**</u> Determine the determinant of the matrices:

$$T = \begin{pmatrix} 1 & -2 & -4 \\ 2 & -3 & 3 \\ 1 & 0 & -3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 3 \\ -2 & 5 & -2 \\ 3 & 1 & -6 \end{pmatrix}$$

$$|T| = 1(9 - 0) - (-2)(-6 - 3) + (-4)(0 + 3) = 9 - 18 - 12 = -21$$

$$|A| = 1(-30 + 2) - 1(12 + 6) + 3(-2 - 15) = -28 - 18 - 51 = -97$$

DETERMINANTS

Determinant of order 2

Consider a 2×2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Determinant of A, denoted $|A|$, is a number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of order 2

- easy to remember (for order 2 only)..

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

DETERMINANTS

- The following properties are true for determinants of *any* order.

- 1. If every element of a row (column) is zero, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1\times 0 - 2\times 0 = 0$, then $|A| = 0$.

2. $|A^T| = |A|$ ← determinant of a matrix = that of its transpose

3. $|AB| = |A||B|$

DETERMINANTS

- Example: Show that the determinant of any orthogonal matrix is either $+1$ or -1.

- For any orthogonal matrix, $AA^T = I$.

- Since $|AA^T| = |A||A^T| = 1$ and $|A^T| = |A|$, so $|A|^2 = 1$

or

- $|A| = \pm 1$.

DETERMINANTS

For any 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Its inverse can be written as $A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Example: Find the inverse of $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

The determinant of A is -2

Hence, the inverse of A is $A^{-1} = \begin{bmatrix} -1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

How to find an inverse for a 3x3 matrix?

DETERMINANTS OF ORDER 3

Consider an example: $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$

Its determinant can be obtained by:

$$|A|=\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}=3\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix}-6\begin{vmatrix}1 & 2\\ 7 & 8\end{vmatrix}+9\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}$$

$$=3(-3)-6(-6)+9(-3)=0$$

You are encouraged to find the determinant by using other rows or columns

INVERSE OF A MATRIX

Inverse of a 2 × 2 Matrix

The inverse of a 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If the determinant is 0, is undefined. So a matrix with a determinant of 0 has no inverse. It is called a *singular* matrix.

Example 1: Finding the Inverse of a Matrix

Find the inverse of the matrix if it is defined.

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

First, check that the determinant is nonzero.
4(1) – 2(3) = 4 – 6 = –2. The determinant is –2, so the matrix has an inverse.

The inverse of $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ is $A^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$.

Example 2: Finding the Inverse of a Matrix

Find the inverse of the matrix if it is defined.

$$\begin{bmatrix} 4 & -3 \\ -\frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

The determinant is, $4\left(\frac{1}{4}\right) - \left(-\frac{1}{3}\right)(-3) = 0$, so *B* has no inverse.

INVERSE OF A 3×3 MATRIX

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

The cofactor for each element of matrix *A*:

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

INVERSE OF A 3×3 MATRIX

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is then given by:

$$\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

INVERSE OF A 3×3 MATRIX

Inverse matrix of $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 1 & 0 & 6\end{bmatrix}$ is given by:

$$A^{-1}=\frac{1}{|A|}\begin{bmatrix}24 & 5 & -4\\ -12 & 3 & 2\\ -2 & -5 & 4\end{bmatrix}^{T}=\frac{1}{22}\begin{bmatrix}24 & -12 & -2\\ 5 & 3 & -5\\ -4 & 2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}12/11 & -6/11 & -1/11\\ 5/22 & 3/22 & -5/22\\ -2/11 & 1/11 & 2/11\end{bmatrix}$$

Inverse Matrix:

If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}.

$$A \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

So, $AB = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Similarly $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Thus B is the inverse of A. In other words $B = A^{-1}$ and A is the inverse of B. ie. $A = B^{-1}$

Question: Find the inverse of the following matrix and show that $A.A^{-1} = A^{-1}.A = I$.

$$A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$$

Solution: Given, $A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

So, $|A| = 1.(0+1) - 4.(6-2) + 4.(-3-0) = 1 - 16 - 12 = -27$

Cofactors:

$A_{11} = (0+1) = 1$

$A_{12} = -(6-2) = -4$

$A_{13} = (-3-0) = -3$

$A_{21} = -(8+4) = -12$

$A_{22} = (2-8) = -6$

$A_{23} = -(-1-8) = 9$

$A_{31} = (4-0) = 4$

$A_{32} = -(1-12) = 11$

$A_{33} = (0-12) = -12$

Now, $Adj(A) = \begin{pmatrix} 1 & -4 & -3 \\ -12 & -6 & 9 \\ 4 & 11 & -12 \end{pmatrix}^T = \begin{pmatrix} 1 & -12 & 4 \\ -4 & -6 & 11 \\ -3 & 9 & -12 \end{pmatrix}$

So, $A^{-1} = \frac{1}{|A|}.Adj(A) = \frac{1}{-27}\begin{pmatrix} 1 & -12 & 4 \\ -4 & -6 & 11 \\ -3 & 9 & -12 \end{pmatrix}$

2nd part: $A.A^{-1} = A \cdot A^{-1} = \frac{1}{-27}\begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -12 & 4 \\ -4 & -6 & 11 \\ -3 & 9 & -12 \end{pmatrix}$

$= \frac{1}{-27}\begin{pmatrix} 1-16-12 & -12-24+36 & 4+44-48 \\ 3-0-3 & -36-0+9 & 12+0-12 \\ 2+4-6 & -24+6+18 & 8-11-24 \end{pmatrix}$

$= \frac{1}{-27}\begin{pmatrix} -27 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

Question: If $P = \begin{pmatrix} 1 & -1 & 4 \\ -4 & -6 & 1 \\ -3 & 1 & -1 \end{pmatrix}$, then find P^{-1} and show that $PP^{-1} = I$

Matrices and Systems of Equations

Week 5
Topics: System of Linear Equations
Page no (60-67)

SYSTEMS OF LINEAR EQUATIONS

- You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution.

- If the coefficient matrix A of a *square* system (a system that has the same

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by

$$X = A^{-1}B.$$

SYSTEMS OF LINEAR EQUATIONS

- The formula $X = A^{-1}B$ is used on most graphing calculators to solve linear systems that have invertible coefficient matrices.

- That is, you enter the $n \times n$ coefficient matrix $[A]$ and the $n \times 1$ column matrix $[B]$. The solution X is given by $[A]^{-1}[B]$.

EXAMPLE 5 – *SOLVING A SYSTEM OF EQUATIONS USING AN INVERSE*

- Use an inverse matrix to

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

- Solution:
- Begin by writing

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

EXAMPLE 5 – *SOLUTION*

cont'd

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

- Then, use Gauss-Jordan elimination to find A^{-1}.

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

- Finally, multiply B by A^{-1} on the
- $X = A^{-1}B$

$$= \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

cont'd

EXAMPLE 5 – *SOLUTION*

- So, the solution is
- $x = 2$, $y = -1$, and $z = -2$.

Solving system of linear equations using matrix method:

Suppose, a system of linear equations are,

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

We can write this as $AX = b \dots\dots\dots\dots\dots.(1)$, where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

From (1) we can write, $X = A^{-1}.b$…………….(2).

Question-1: Solve the following system of linear equations using matrix method.

$$2x - 2y + z = 1$$
$$x + 3y - 2z = 1$$
$$3x - y - z = -2$$

Given system of equations can be written as $AX = b$ ……(2), where

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 3 & -2 \\ 3 & -1 & -1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

From (2) we can write $X = A^{-1}b$ ………..(3)

Now, $|A| = 2(-3-2) + 2(-1+6) + 1(-1-9) = -10$

Cofactors:

A_{11}=$(-3-2) = -5$

$A_{12} = -(-1+6) = -5$

$A_{13} = (-1-9) = -10$

$A_{21} = -(2+1) = -3$

$A_{22} = (-2-3) = -5$

$A_{23} = -(-2+6) = -4$

$A_{31} = (4-3) = 1$

$A_{32} = -(-4-1) = 5$

$A_{33} = (6+2) = 8$

So, $Adj A = \begin{pmatrix} -5 & -5 & -10 \\ -3 & -5 & -4 \\ 1 & 5 & 8 \end{pmatrix}^T = \begin{pmatrix} -5 & -3 & 1 \\ -5 & -5 & 5 \\ -10 & -4 & 8 \end{pmatrix}$

$So, A^{-1} = \frac{1}{|A|}.Adj(A) = \frac{1}{-10}\begin{pmatrix} -5 & -3 & 1 \\ -5 & -5 & 5 \\ -10 & -4 & 8 \end{pmatrix}$

From (3) we get, $X = \frac{1}{-10}\begin{pmatrix} -5 & -3 & 1 \\ -5 & -5 & 5 \\ -10 & -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -5-3-2 \\ -5-5-10 \\ -10-4-16 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -10 \\ -20 \\ -30 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Solution of the system $x = 1,\ y = 2,\ z = 3$

<u>Question-2:</u> Solve the following system of linear equations using matrix method.

$(i)\quad \begin{aligned} x+2y+z&=3 \\ 2x+3y+z&=4 \\ 3x-y-z&=4 \end{aligned} \qquad (ii)\quad \begin{aligned} x-2y+3z&=7 \\ 2x+y-z&=1 \\ x-y-z&=-6 \end{aligned}$

CRAMER'S RULE

GABRIEL CRAMER WAS A SWISS MATHEMATICIAN (1704-1752)

Week 6
Topics: Cramer's rule

Page no. (68-90)

INTRODUCTION

- Cramer’s Rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.
- Cramer’s Rule relies on determinants

COEFFICIENT MATRICES

- You can use determinants to solve a system of linear equations.
- You use the coefficient matrix of the linear system.
- **Linear System** **Coeff Matrix** ax+by=e
  cx+dy=f

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

CRAMER'S RULE FOR 2X2 SYSTEM

- Let A be the coefficient matrix
- **Linear System** **Coeff Matrix**
- ax+by=e
- cx+dy=f

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- If detA ≠ 0, then the system has exactly one solution:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

KEY POINTS

- The denominator consists of the coefficients of variables (x in the first column, and y in the second column).
- The numerator is the same as the denominator, with the constants replacing the coefficients of the variable for which you are solving.

EXAMPLE - APPLYING CRAMER'S RULE ON A SYSTEM OF TWO EQUATIONS

Solve the system:

- 8x+5y= 2
- 2x-4y= -10

The coefficient matrix is: $\begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$ and $\begin{vmatrix} 8 & 5 \\ 2 & -4 \end{vmatrix} = (-32) - (10) = -42$

So: $x = \dfrac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42}$ and $y = \dfrac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42}$

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42} = \frac{-8-(-50)}{-42} = \frac{42}{-42} = -1$$

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42} = \frac{-80-4}{-42} = \frac{-84}{-42} = 2$$

Solution: (-1,2)

APPLYING CRAMER'S RULE ON A SYSTEM OF TWO EQUATIONS

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

$$D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D}$$

$$\begin{cases} 2x - 3y = -16 \\ 3x + 5y = 14 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix} = (2)(5) - (-3)(3) = 10 + 9 = 19$$

$$D_x = \begin{vmatrix} -16 & -3 \\ 14 & 5 \end{vmatrix} = (-16)(5) - (-3)(14) = -80 + 42 = -38$$

$$D_y = \begin{vmatrix} 2 & -16 \\ 3 & 14 \end{vmatrix} = (2)(14) - (3)(-16) = 28 + 48 = 76$$

$$x = \frac{D_x}{D} = \frac{-38}{19} = -2 \quad y = \frac{D_y}{D} = \frac{76}{19} = 4$$

EVALUATING A 3X3 DETERMINANT
(EXPANDING ALONG THE TOP ROW)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

- Expanding by Minors (little 2x2 determinants)

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 3 \\ 1 & 2 & 3 \end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} - (3)\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (1)(-6) \quad -(3)(3) \quad +(-2)(4)$$

$$= -6 \quad -9 \quad -8 = -23$$

USING CRAMER'S RULE TO SOLVE A SYSTEM OF THREE EQUATIONS

Consider the following set of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

USING CRAMER'S RULE TO SOLVE A SYSTEM OF THREE EQUATIONS

The system of equations above can be written in a matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

USING CRAMER'S RULE TO SOLVE A SYSTEM OF THREE EQUATIONS

Define

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ and \ [B] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $D \neq 0$, then the system has a unique solution as shown below (Cramer's Rule).

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

USING CRAMER'S RULE TO SOLVE A SYSTEM OF THREE EQUATIONS

where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{vmatrix} \qquad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{12} & b_2 & a_{23} \\ a_{13} & b_3 & a_{33} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ a_{13} & a_{32} & b_3 \end{vmatrix}$$

EXAMPLE 1

Consider the following equations:

$$2x_1 - 4x_2 + 5x_3 = 36$$
$$-3x_1 + 5x_2 + 7x_3 = 7$$
$$5x_1 + 3x_2 - 8x_3 = -31$$
$$[A][x] = [B]$$

where

$$[A] = \begin{bmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{bmatrix}$$

EXAMPLE 1

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad and \quad [B] = \begin{bmatrix} 36 \\ 7 \\ -31 \end{bmatrix}$$

$$D = \begin{vmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{vmatrix} = -336$$

$$D_1 = \begin{vmatrix} 36 & -4 & 5 \\ 7 & 5 & 7 \\ -31 & 3 & -8 \end{vmatrix} = -672$$

EXAMPLE 1

$$D_2 = \begin{vmatrix} 2 & 36 & 5 \\ -3 & 7 & 7 \\ 5 & -31 & -8 \end{vmatrix} = 1008$$

$$D_3 = \begin{vmatrix} 2 & -4 & 36 \\ -3 & 5 & 7 \\ 5 & 3 & -31 \end{vmatrix} = -1344$$

$$x_1 = \frac{D_1}{D} = \frac{-672}{-336} = 2$$

$$x_2 = \frac{D_2}{D} = \frac{1008}{-336} = -3$$

$$x_3 = \frac{D_3}{D} = \frac{-1344}{-336} = 4$$

CRAMER'S RULE - 3 X 3

- Consider the 3 equation system below with variables *x*, *y* and *z*:

$$
\begin{aligned}
a_1x + b_1y + c_1z &= C_1 \\
a_2x + b_2y + c_2z &= C_2 \\
a_3x + b_3y + c_3z &= C_3
\end{aligned}
$$

CRAMER'S RULE - 3 X 3

- The formulae for the values of *x*, *y* and *z* are shown below. Notice that all three have the same denominator.

$$x = \frac{\begin{vmatrix} C_1 & b_1 & c_1 \\ C_2 & b_2 & c_2 \\ C_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & C_1 & c_1 \\ a_2 & C_2 & c_2 \\ a_3 & C_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & C_1 \\ a_2 & b_2 & C_2 \\ a_3 & b_3 & C_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

EXAMPLE 1

- Solve the system : $3x - 2y + z = 9$
- $x + 2y - 2z = -5$

  $x + y - 4z = -2$

$$x = \frac{\begin{vmatrix} 9 & -2 & 1 \\ -5 & 2 & -2 \\ -2 & 1 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{-23}{-23} = 1$$

$$y = \frac{\begin{vmatrix} 3 & 9 & 1 \\ 1 & -5 & -2 \\ 1 & -2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{69}{-23} = -3$$

EXAMPLE 1

$$z = \frac{\begin{vmatrix} 3 & -2 & 9 \\ 1 & 2 & -5 \\ 1 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{0}{-23} = 0$$

The solution is

(1, -3, 0)

CRAMER'S RULE

- Not all systems have a definite solution. If the determinant of the coefficient matrix is zero, a solution cannot be found using Cramer's Rule because of division by zero.
- When the solution cannot be determined, one of two conditions exists:
  - The planes graphed by each equation are parallel and there are no solutions.
  - The three planes share one line (like three pages of a book share the same spine) or represent the same plane, in which case there are infinite solutions.

Suppose, a system of linear equations are,

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

We can write this as $AX = b$ … … … … … . (1), where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Now $$A_x = \begin{pmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & c_2 & c_3 \end{pmatrix},\quad A_y = \begin{pmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ c_1 & d_3 & c_3 \end{pmatrix},\quad A_z = \begin{pmatrix} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ c_1 & c_2 & d_3 \end{pmatrix}$$

According to Cramer's rule, $x = \frac{|A_x|}{|A|}, y = \frac{|A_y|}{|A|},\ z = \frac{|A_z|}{|A|},$

Question-1: Solve the following system of linear equations using Cramer's rule.

$$(i)\quad \begin{aligned} x+2y+z&=3 \\ 2x+3y+z&=3 \\ 3x-y-z&=-6 \end{aligned} \qquad\qquad (ii)\quad \begin{aligned} x-2y+3z&=7 \\ 2x+y-z&=1 \\ x-y-z&=-6 \end{aligned}$$

Solution: **(i)**

We can write this as $AX = b$ … … … … … . (1), where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & -1 & -1 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix}$$

Now $$A_x = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ -6 & -1 & -1 \end{pmatrix},\quad A_y = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 3 & 1 \\ 3 & -6 & -1 \end{pmatrix},\quad A_z = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & -1 & -6 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & -1 & -1 \end{vmatrix} = -3\,, \quad |A_x| = \begin{vmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ -6 & -1 & -1 \end{vmatrix} = 3,$$

$$|A_y| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 3 & 1 \\ 3 & -6 & -1 \end{vmatrix} = -3, \quad |A_z| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & -1 & -6 \end{vmatrix} = -6$$

According to Cramer's rule, $x = \frac{|A_x|}{|A|} = -1\,, y = \frac{|A_y|}{|A|} = 1,\ z = \frac{|A_z|}{|A|} = 2$

Week 7
Topics : Rank of a matrix
Page no (90-96)

Rank of a Matrix

The maximum number of linearly independent rows in a matrix *A* is called the row rank of *A*, and the maximum number of linearly independent columns in *A* is called the column rank of *A*. If *A* is an *m* by *n* matrix, that is, if *A* has *m* rows and *n* columns, then it is obvious that

$$\left.\begin{array}{r} \boldsymbol{Raw\ rank\ of\ A \leq m} \\ \boldsymbol{Column\ rank\ \ of\ A \leq n} \end{array}\right\} \qquad \cdots\cdots(\mathbf{1})$$

However, is that for any matrix *A*, **the row rank of *A* = the column rank of *A***

Because of this fact, there is no reason to distinguish between row rank and column rank; the common value is simply called the **rank** of the matrix. Therefore, if *A* is *m x n*, it follows from the inequalities in (1) that

$$\boldsymbol{rank(\ A_{m\times n}) \leq min(m, n)} \qquad \cdots\cdots(\mathbf{2})$$

RANK OF A MATRIX

- The rank of a matrix is equal to the number of linearly independent rows (or columns) in it. Hence, it cannot more than its number of rows and columns. For example, if we consider the identity matrix of order 3 × 3, all its rows (or columns) are linearly independent and hence its rank is 3.

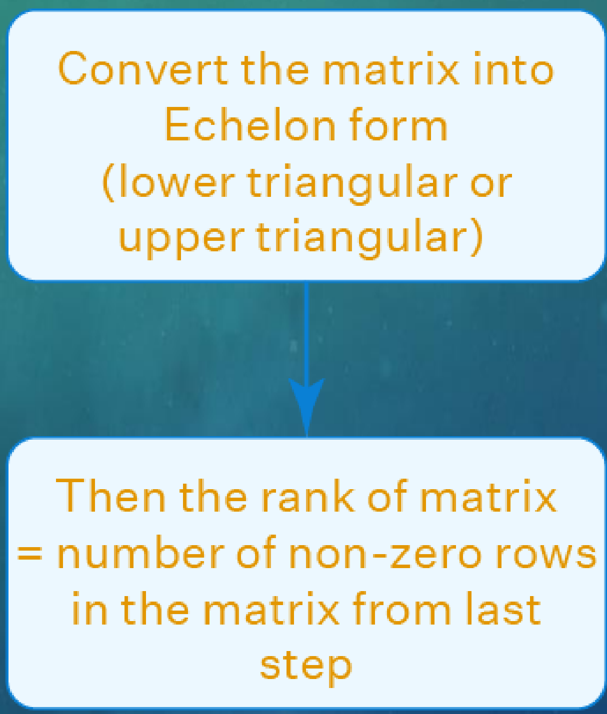
HOW TO FIND THE RANK OF A MATRIX?

- The rank of a matrix can be found using three methods. The most easiest of these methods is "converting matrix into echelon form".
- Minor method
- Using echelon form
- Using normal form
- Let us study each of these methods in detail.

Finding Rank of a Matrix by Minor Method

- Here are the steps to find the rank of a matrix A by the minor method.
- Find the determinant of A (if A is a square matrix). If det (A) ≠ 0, then the rank of A = order of A.
- If either det A = 0 (in case of a square matrix) or A is a rectangular matrix, then see whether there exists any minor of maximum possible order is non-zero. If there exists such non-zero minor, then rank of A = order of that particular minor.
- Repeat the above step if all the minors of the order considered in the above step are zeros and then try to find a non-zero minor of order that is 1 less than the order from the above step.
- Here is an example.
- Example: Find the rank of the matrix ρ (A) if A = .
- Solution:
- A is a square matrix and so we can find its determinant.
- det (A) = 1 (45 - 48) - 2 (36 - 42) + 3 (32 - 35)
= -3 + 12 - 9
= 0
- So ρ (A) ≠ order of the matrix. i.e., ρ (A) ≠ 3.
- Now, we will see whether we can find any non-zero minor of order 2.
- = 5 - 8 = -3 ≠ 0.
- So there exists a minor of order 2 (or 2 × 2) which is non-zero. So the rank of A, ρ (A) = 2.

Finding Rank of a Matrix Using Echelon Form



Example-1: Find the rank of the matrix $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$

Solution:

First, because the matrix is 4×3, its rank can be no greater than 3. Therefore, at least one of the four rows will become a row of zeros. Perform the following row operations:

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \xrightarrow[r_4'=-r_1+r_4]{r_2'=-2r_1+r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow[r_4'=r_2+r_4]{r_3'=2r_2+r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[r_4'=-2r_3+r_4]{r_2'=-r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_3'=-r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are 3 nonzero rows remaining in this echelon form of *B*, rank(B)=3

Example-2: Determine the rank of the following matrices.

$$C = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$$

Question: Express M as a linear combination of the matrices A, B and C where

$M = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$

Solution: Given,

$$M = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$$

Here, M will be a linear combination of A, B, C if there exist some scalars a_1, a_2, a_3 such that

$a_1.A + a_2.B + a_3.C = M$ ………………….(1)

$$\text{Or,}\quad a_1.\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + a_2.\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + a_3.\begin{bmatrix}1 & 1\\4 & 5\end{bmatrix} = \begin{bmatrix}4 & 7\\7 & 9\end{bmatrix}$$

$$\text{Or}\quad \begin{bmatrix}a_1 & a_1\\a_1 & a_1\end{bmatrix} + \begin{bmatrix}a_2 & 2a_2\\3a_2 & 4a_2\end{bmatrix} + \begin{bmatrix}a_3 & a_3\\4a_3 & 5a_3\end{bmatrix} = \begin{bmatrix}4 & 7\\7 & 9\end{bmatrix}$$

$$\text{Or}\quad \begin{bmatrix}a_1 + a_2 + a_3 & a_1 + 2a_2 + a_3\\a_1 + 3a_2 + 4a_3 & a_1 + 4a_2 + 5a_3\end{bmatrix} = \begin{bmatrix}4 & 7\\7 & 9\end{bmatrix}$$

$$a_1 + a_2 + a_3 = 4$$

$$a_1 + 2a_2 + a_3 = 7$$

$$a_1 + 3a_2 + 4a_3 = 7$$

$$a_1 + 4a_2 + 5a_3 = 9$$

Solving the above system of equations, we get, $a_1 = 2,\ a_2 = 3,\ a_3 = -1.$

From (i), we get, $2A + 3B - C = M$ or, $M = 2A + 3B - C$. (Expressed)

Week 8

Topics: Eigenvalue and Eigenvector Cayley Hamilton Theorem

Page no (96-112)

Eigenvalue Definition

Eigenvalues are the special set of scalars associated with the system of linear equations. It is mostly used in matrix equations. ‘Eigen’ is a German word that means ‘proper’ or ‘characteristic’.

Therefore, the term eigenvalue can be termed as characteristic value, characteristic root, proper values or latent roots as well. In simple words, the eigenvalue is a scalar that is used to transform the eigenvector. The basic equation is

Ax = λx

The number or scalar value "**λ**" is an eigenvalue of A.

In Mathematics, an eigenvector corresponds to the real non zero eigenvalues which point in the direction stretched by the transformation whereas eigenvalue is considered as a factor by which it is stretched. In case, if the eigenvalue is negative, the direction of the transformation is negative.

For every real matrix, there is an eigenvalue. Sometimes it might be complex. The existence of the eigenvalue for the complex matrices is equal to the fundamental theorem of algebra.

What are Eigenvectors?

Eigenvectors are the vectors (non-zero) that do not change the direction when any linear transformation is applied. It changes by only a scalar factor. In a brief, we can say, if A is a linear transformation from a vector space V and **x** is a vector in V, which is not a zero vector, then v is an eigenvector of A if A(X) is a scalar multiple of **x**.

An **Eigenspace** of vector **x** consists of a set of all eigenvectors with the equivalent eigenvalue collectively with the zero vector. Though, the zero vector is not an eigenvector.

Let us say A is an "n × n" matrix and λ is an eigenvalue of matrix A, then **x**, a non-zero vector, is called as eigenvector if it satisfies the given below expression;

A**x** = λ**x**

x is an eigenvector of A corresponding to eigenvalue, λ.

Note:

- There could be infinitely many Eigenvectors, corresponding to one eigenvalue.
- For distinct eigenvalues, the eigenvectors are linearly dependent.

EIGENVALUES & EIGENVECTORS, DIAGONALISATION & THE CAYLEY-HAMILTON THEOREM

Before you start:

- You need to be able to find the determinant and the inverse of a 2x2 matrix and 3x3 matrix.
- You need to be familiar with the work on matrices on FP1, in particular you should know what is meant by an invariant line.

SOLVING SIMULTANEOUS EQUATIONS USING MATRICES, EIGENVALUES & EIGENVECTORS, CAYLEY-HAMILTON THEOREM

When you have finished…
You should:

- Understand the meaning of eigenvalue and eigenvector, and be able to find these for 2 x 2 or 3 x 3 matrices whenever this is possible.
- Understand the term characteristic equation of a 2 x 2 or 3 x 3 matrix.
- Be able to form the matrix of eigenvectors and use this to reduce a matrix to diagonal form.
- Be able to find use the diagonal form to find powers of a 2 x 2 or 3 x 3 matrix.
- Understand that every 2 x 2 or 3 x 3 matrix satisfies its own characteristic equation, and be able to use this.

RECAP: FP1 INVARIANT LINES

A reflection in two dimensions has a **line of invariant points**. In FP1 we learned how to find these by solving the matrix equation $\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

A line of invariant points is a special case of an **invariant line**, the points on which may move but will stay on the same line. Examples are lines through the origin under an enlargement with centre the origin.

An important result which should not be forgotten is that any transformation represented by a matrix fixes the origin.

A transformation represented by a 2 × 2 matrix maps lines to lines

The next concept we will consider is which lines map onto themselves under a transformation (represented by a matrix)

BEFORE WE START...

<u>Terminology</u>
If s is a non-zero vector such that **M**s = λs
s is called an **eigenvector** of **M**; λ is the corresponding **eigenvalue**.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is an eigenvector of } \mathbf{M} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \text{ because } \mathbf{M}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

BEFORE WE START...

Properties of eigenvectors

1. For a given eigenvalue, the eigenvector is not unique.

   E.g. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{M} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ because $\mathbf{M}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   But also $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

2. Under the transformation, the eigenvector is enlarged by a scale factor equal to its eigenvalue.
3. The transformation does not change the direction of the eigenvector.
4. $\lambda = 0$ is a possible eigenvalue.

BEFORE WE START...

<u>Finding eigenvectors</u>

We need to solve the equation $\mathbf{Ms} = \lambda\mathbf{s}$ to find $\mathbf{s}$.

But $$\mathbf{Ms} = \lambda\mathbf{s}$$

$$\Leftrightarrow \quad \mathbf{Ms} - \lambda\mathbf{s} = \mathbf{0}$$

$$\Leftrightarrow \quad \mathbf{Ms} - \lambda\mathbf{Is} = \mathbf{0}$$

$$\Leftrightarrow \quad (\mathbf{M} - \lambda\mathbf{I})\mathbf{s} = \mathbf{0}$$

If $\mathbf{M} - \lambda\mathbf{I}$ is non-singular, this equation has a solution

$$\mathbf{s} = (\mathbf{M} - \lambda\mathbf{I})^{-1}\mathbf{0} = \mathbf{0}$$

but we are seeking non-zero solutions for $\mathbf{s}$.

For these to exist, we require $\mathbf{M} - \lambda\mathbf{I}$ to be singular.

$$\Leftrightarrow \quad \det(\mathbf{M} - \lambda\mathbf{I}) = 0$$

This is called the **characteristic equation** of $\mathbf{M}$.

EIGENVALUES AND EIGENVECTORS OF A 2×2 MATRIX

<u>Example</u>

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

characteristic equation of $\mathbf{M}$
$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

For each eigenvalue λ, find a corresponding eigenvector **s** by solving $(\mathbf{M} - \lambda\mathbf{I})\mathbf{s} = \mathbf{0}$ for non-zero **s**.

$$\mathbf{M} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}$$

$\lambda = 2$

$\lambda = 3$

EIGENVALUES AND EIGENVECTORS OF 3×3 MATRICES

The theory is exactly the same for 3 × 3 matrices.

Example

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 6 \\ 3 & -3 & 27 \\ 1 & -1 & 7 \end{bmatrix}$.

Form the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

$$\mathbf{M} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & -1 & 6 \\ 3 & -3-\lambda & 27 \\ 1 & -1 & 7-\lambda \end{bmatrix}$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (2-\lambda)\begin{vmatrix} -3-\lambda & 27 \\ -1 & 7-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & 27 \\ 1 & 7-\lambda \end{vmatrix} + 6\begin{vmatrix} 3 & -3-\lambda \\ 1 & -1 \end{vmatrix} = 0$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (2-\lambda)\begin{vmatrix} -3-\lambda & 27 \\ -1 & 7-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & 27 \\ 1 & 7-\lambda \end{vmatrix} + 6\begin{vmatrix} 3 & -3-\lambda \\ 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)(\lambda^2 - 4\lambda + 6) + 1(-3\lambda - 6) + 6\lambda = 0$$
$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$
$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Solve the characteristic equation.

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

eigenvalues are 1, 2 and 3.

For each eigenvalue λ, find a corresponding eigenvector **s** by solving $(\mathbf{M} - \lambda\mathbf{I})\mathbf{s} = \mathbf{0}$ for non-zero **s**.

$\lambda = 1$

$(\mathbf{M} - \lambda\mathbf{I})\mathbf{s} = \mathbf{0}$

$$\begin{bmatrix} 1 & -1 & 6 \\ 3 & -4 & 27 \\ 1 & -1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ (1) $x - y + 6z = 0$

(2) $3x - 4y + 27z = 0$

(3) $x - y + 6z = 0$

$\Rightarrow$ (1) = (3)

(1) × 3 $3x - 3y + 18z = 0$

– (2): $y - 9z = 0 \Rightarrow y = 9z$

$\Rightarrow$ in (1); $x - 9z + 6z = 0 \Rightarrow x = 3z$

Let $z = 1$ (why not?): $x = 3$ and $y = 9$, so an eigenvector is $\begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$

$\lambda = 2$: an eigenvector is $\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$

$\lambda = 3$: an eigenvector is $\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$

For a 3×3 matrix, the characteristic equation is a cubic. This will always have at least one real root so there will always be at least one real eigenvalue and associated invariant line.

The situation where we do not have three real distinct eigenvalues is beyond the specification, but is explored in the example on p.109.

(useful to remember)

Sum of roots = Tr(**M**)
Product of roots = det(**M**)

For a 2 × 2 matrix $\mathbf{M} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, the characteristic equation is

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0 \Rightarrow (a - \lambda)(d - \lambda) - bc = 0$$
$$\Rightarrow \quad \lambda^2 - (a + d)\lambda + ad - bc = 0$$
$$\Rightarrow \quad \lambda^2 - \mathrm{Tr}(\mathbf{M})\lambda + \det(\mathbf{M}) = 0$$

These properties also hold for $n \times n$ matrices.

Eigenvalues of a Square Matrix

Suppose, $A_{n\times n}$ is a square matrix, then [A- λI] is called an Eigen or characteristic matrix, which is an indefinite or undefined scalar. Where determinant of Eigen matrix can be written as, **|A- λI| and |A- λI| = 0** is the Eigen equation or characteristics equation, where "I" is the identity matrix. The roots of an Eigen matrix are called Eigen roots.

Eigenvalues of a triangular matrix and diagonal matrix are equivalent to the elements on the principal diagonals. But eigenvalues of the scalar matrix are the scalar only.

Question: Find the eigenvalue of the given matrix $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$

Solution: The characteristic equation of A is $|\lambda I - A| = 0$

$$\Rightarrow \left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-4)(\lambda^2-9+8)+6(-\lambda-3+2)-6(-4-\lambda+3)=0$$

$$\Rightarrow (\lambda-4)(\lambda^2-1)-6\lambda-6+6\lambda+6=0$$

$$\Rightarrow (\lambda-4)(\lambda^2-1)=0$$

$$\Rightarrow (\lambda-4)=0, \quad (\lambda^2-1)=0$$

$$\Rightarrow \lambda=4, \quad \lambda=\pm\sqrt{1}=\pm1$$

$$\therefore \lambda = 4,-1,1$$

Therefore, the eigenvalues of A are $\lambda = -1, 1, 4$

Exercise: Find the eigenvalue of the given matrix $B = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Question: Find the characteristic equation of the following matrices and verify Cayley-Hamilton theorem for it. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

<u>Solution:</u> The characteristic equation of A is $|\lambda I - A| = 0$

$$\Rightarrow \left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -2 & -3 \\ -2 & \lambda + 1 & -1 \\ -3 & -1 & \lambda - 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 1 - 1) + 2(-2\lambda + 2 - 3) - 3(2 + 3\lambda + 3) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 2) - 4\lambda - 2 - 9\lambda - 15 = 0$$

$$\Rightarrow \lambda^3 - 2\lambda - \lambda^2 + 2 - 13\lambda - 17 = 0$$

$$\therefore \quad \lambda^3 - \lambda^2 - 15\lambda - 15 = 0$$

Now in order to verify Cayley-Hamilton theorem we have to show that $A^3 - A^2 - 15A - 15I = 0$

Here, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ $A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{pmatrix}$ and

$$A^3 = A \cdot A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{pmatrix} = \begin{pmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{pmatrix}$$

$$A^3 - A^2 - 15A - 15I$$

$$= \begin{pmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{pmatrix} - \begin{pmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{pmatrix} - 15\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix} - 15\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 44-14-15-15 & 33-3-30-0 & 53-8-45-0 \\ 33-3-30-0 & 6-6+15-15 & 21-6-15-0 \\ 53-8-45-0 & 21-6-15-0 & 41-11-15-15 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the Cayley-Hamilton theorem is verified.

Week 9

Topics: Application of Linear algebra

Page no (114-136)

1. Find Inverse of matrix using Cayley Hamilton method

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution:

To apply the Cayley-Hamilton theorem, we first determine the characteristic polynomial p(t) of the matrix A.

$|A - tI|$

$$= \begin{vmatrix} (3 - t) & 1 & 1 \\ -1 & (2 - t) & 1 \\ 1 & 1 & (1 - t) \end{vmatrix}$$

$$= (3 - t)((2 - t) \times (1 - t) - 1 \times 1) - 1((-1) \times (1 - t) - 1 \times 1) + 1((-1) \times 1 - (2 - t) \times 1)$$

$$= (3 - t)\left(\left(2 - 3t + t^2\right) - 1\right) - 1((-1 + t) - 1) + 1((-1) - (2 - t))$$

$$= (3 - t)\left(1 - 3t + t^2\right) - 1(-2 + t) + 1(-3 + t)$$

$$= \left(3 - 10t + 6t^2 - t^3\right) - (-2 + t) + (-3 + t)$$

$$= -t^3 + 6t^2 - 10t + 2$$

$$p(t) = -t^3 + 6t^2 - 10t + 2$$

The Cayley-Hamilton theorem yields that

$$O = p(A) = -A^3 + 6A^2 - 10A + 2I$$

Rearranging terms, we have

$$\therefore 2I = A^3 - 6A^2 + 10A$$

$$\therefore 2I = A\left(A^2 - 6A + 10I\right)$$

$$\therefore A^{-1} = \frac{1}{2}\left(A^2 - 6A + 10I\right)$$

Now, first we find $A^2 - 6A + 10I$

$$A^2 = A \times A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 3 + 1 \times -1 + 1 \times 1 & 3 \times 1 + 1 \times 2 + 1 \times 1 & 3 \times 1 + 1 \times 1 + 1 \times 1 \\ -1 \times 3 + 2 \times -1 + 1 \times 1 & -1 \times 1 + 2 \times 2 + 1 \times 1 & -1 \times 1 + 2 \times 1 + 1 \times 1 \\ 1 \times 3 + 1 \times -1 + 1 \times 1 & 1 \times 1 + 1 \times 2 + 1 \times 1 & 1 \times 1 + 1 \times 1 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 1 + 1 & 3 + 2 + 1 & 3 + 1 + 1 \\ -3 - 2 + 1 & -1 + 4 + 1 & -1 + 2 + 1 \\ 3 - 1 + 1 & 1 + 2 + 1 & 1 + 1 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 5 \\ -4 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 9 & 6 & 5 \\ -4 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

$$6 \times A = 6 \times \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 6 \\ -6 & 12 & 6 \\ 6 & 6 & 6 \end{bmatrix}$$

$$A^2 - 6 \times A = \begin{bmatrix} 9 & 6 & 5 \\ -4 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix} - \begin{bmatrix} 18 & 6 & 6 \\ -6 & 12 & 6 \\ 6 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 9+18 & 6+6 & 5+6 \\ -4-6 & 4+12 & 2+6 \\ 3+6 & 4+6 & 3+6 \end{bmatrix} = \begin{bmatrix} -9 & 0 & -1 \\ 2 & -8 & -4 \\ -3 & -2 & -3 \end{bmatrix}$$

$$10 \times I = 10 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$A^2 - 6 \times A + 10 \times I = \begin{bmatrix} -9 & 0 & -1 \\ 2 & -8 & -4 \\ -3 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} -9+10 & 0+0 & -1+0 \\ 2+0 & -8+10 & -4+0 \\ -3+0 & -2+0 & -3+10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -4 \\ -3 & -2 & 7 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{2}\left(A^2 - 6A + 10I\right)$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -4 \\ -3 & -2 & 7 \end{bmatrix}$$

Exercise: Find the characteristic equation of the following matrices and verify Cayley-Hamilton theorem for it.

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 3 \\ -1 & -1 & 0 \end{pmatrix}$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- **Example 2:** Determine if the following system is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned} \qquad \text{----(4)}$$

- **Solution:** The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- To obtain an x_1 in in the first equation, interchange rows 1 and 2.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

- To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$ ----(5)

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- Next, use the x_2 term in the second equation to eliminate the $-(1/2)x_2$ term from the third equation. Add $1/2$ times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{----(6)}$$

- The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation.

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 5/2 \end{aligned} \quad \text{----(7)}$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- The equation $0 = 5/2$ is a short form of

$$0x_1 + 0x_2 + 0x_3 = 5/2.$$

$$0 = 5/2$$

- There are no values of x_1, x_2, x_3 that satisfy (7) because the equation is never true.

- Since (7) and (4) have the same solution set, the original system is inconsistent (*i.e.*, has no solution).

QUIZ 1

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1. Multiply the 2nd row by 8, and add the result to the 3rd row. Write down the answer.
2. Interchange rows 1 and 3 of the original matrix, after multiplying the 1st row by 2.

LINEAR EQUATIONS IN LINEAR ALGEBRA

SYSTEMS OF LINEAR EQUATIONS

LINEAR EQUATION

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where *b* and the coefficients $a_1, \ldots, a_n$ are real or complex numbers that are usually known in advance.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

LINEAR EQUATION

- A **solution** of the system is a list (s_1, s_2,..., s_n) of numbers that makes each equation a true statement when the values s_1,..., s_n are substituted for x_1,..., x_n, respectively.

- The set of all possible solutions is called the **solution set** of the linear system.

- Two linear systems are called **equivalent** if they have the same solution set.

LINEAR EQUATION

- A system of linear equations has
    1. no solution, or
    2. exactly one solution, or
    3. infinitely many solutions.
- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system of linear equation is said to be **inconsistent** if it has no solution.

MATRIX NOTATION

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix.**
- For the following system of equations,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned}$$

the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

is called the **coefficient matrix** of the system.

MATRIX NOTATION

- An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.
- For the given system of equations,

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the augmented matrix.

MATRIX SIZE

- The size of a matrix tells how many rows and columns it has. If *m* and *n* are positive numbers, an $m \times n$ **matrix** is a rectangular array of numbers with *m* rows and *n* columns. (The number of rows always comes first.)
- The basic strategy for solving a linear system is to replace one system with an equivalent system (*i.e.*, one with the same solution set) that is easier to solve.

SOLVING SYSTEM OF EQUATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- Keep x_1 in the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3.

$$\begin{aligned} 4x_1 - 8x_2 + 4x_3 &= 0 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \\ \hline -3x_2 + 13x_3 &= -9 \end{aligned}$$

SOLVING SYSTEM OF EQUATIONS

- The result of this calculation is written in place of the original third equation.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Now, multiply equation 2 by $1/2$ in order to obtain 1 as the coefficient for x_2.

SOLVING SYSTEM OF EQUATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3.

$$\begin{aligned} 3x_2 - 12x_3 &= 12 \\ -3x_2 + 13x_3 &= -9 \\ \hline x_3 &= 3 \end{aligned}$$

SOLVING SYSTEM OF EQUATIONS

- The new system has a triangular form.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ \hline x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Now, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 term in equation 3 first to eliminate the $-4x_3$ and x_3 terms in equations 2 and 1.

SOLVING SYSTEM OF EQUATIONS

- Now, combine the results of these two operations.

$$\begin{array}{r} 4x_3 = 12 \\ x_2 - 4x_3 = 4 \\ \hline x_2 = 16 \end{array} \qquad \begin{array}{r} -x_3 = -3 \\ x_1 - 2x_2 + x_3 = 0 \\ \hline x_1 - 2x_2 = -3 \end{array}$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

$-2x_2$

- Move back to the x_2 in equation 2, and use it to eliminate the above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Thus, the only solution of the original system is (29,16,3). To verify that (29,16,3) is a solution, substitute these values into the left side of the original system, and compute.

$$(29) - 2(16) + (3) = 29 - 32 + 3 = 0$$

$$2(16) - 8(3) = 32 - 24 = 8$$

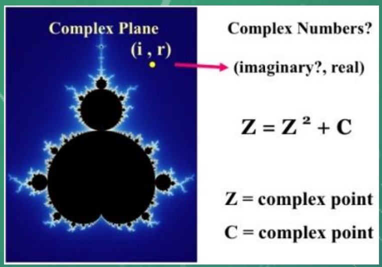
$$-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9$$

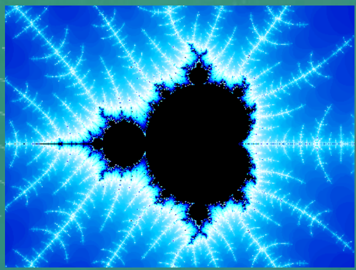
- The results agree with the right side of the original system, so (29,16,3) is a solution of the system.

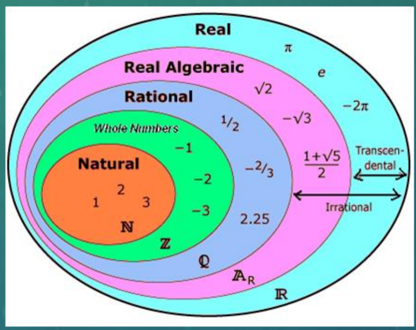
Week 10

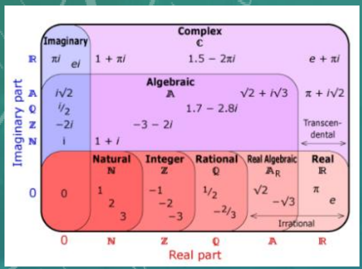
Topics: Complex number

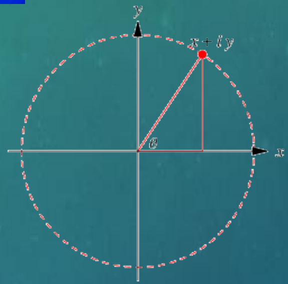
Page no (136-156)



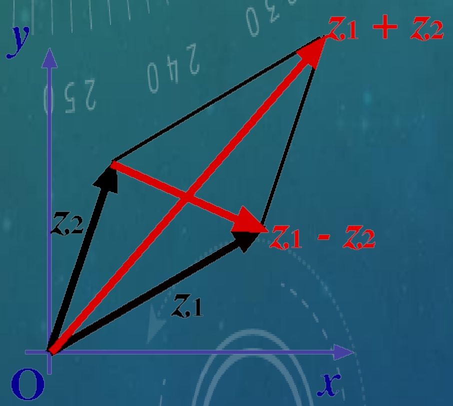


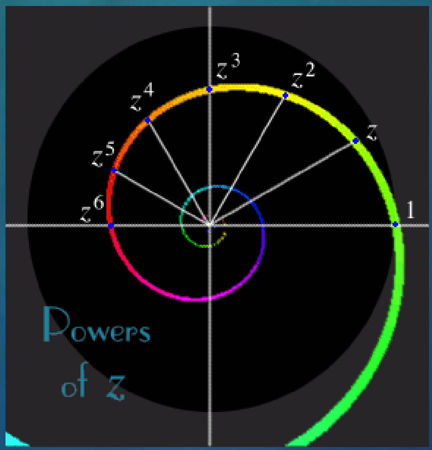


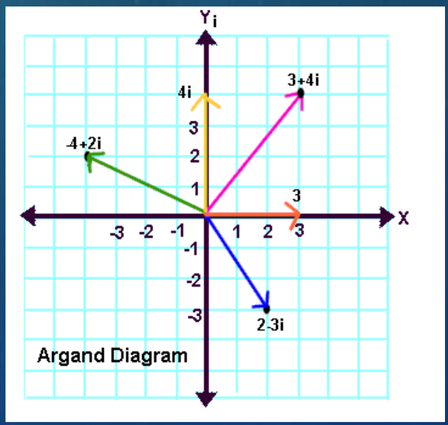




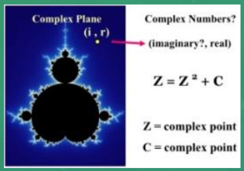
Complex Numbers







COMPLEX NUMBERS



You can use both real and imaginary numbers to solve equations

At GCSE level you met the Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The part under the square root sign is known as the 'discriminant', and can be used to determine how many solutions the equation has:

$$b^2 - 4ac > 0 \quad \longrightarrow \; 2 \; real \; roots$$

$$b^2 - 4ac = 0 \quad \longrightarrow \; 1 \; real \; root$$

$$b^2 - 4ac < 0 \quad \longrightarrow \; 0 \; real \; roots$$

The problem is that we cannot square root a negative number, hence the lack of real roots in the 3rd case above

To solve these equations, we can use the imaginary number 'i'

$$i = \sqrt{-1}$$

The imaginary number 'i' can be combined with real numbers to create 'complex numbers'

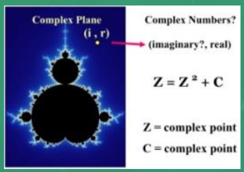
An example of a complex number would be:

$$5 + 2i$$

Complex numbers can be added, subtracted, multiplied and divided in the same way you would with an algebraic expression

1A

COMPLEX NUMBERS



You can use both real and imaginary numbers to solve equations

1) Write √-36 in terms of i

This sign means the **positive** square root →

$\sqrt{-36}$

$\sqrt{36}\sqrt{-1}$

$= 6i$

Split up using surd manipulation

Simplify each part → √-1 = i

2) Write √-28 in terms of i

$\sqrt{-28}$

$\sqrt{28}\sqrt{-1}$

$\sqrt{4}\sqrt{7}\sqrt{-1}$

$= 2\sqrt{7}i$

$= 2i\sqrt{7}$

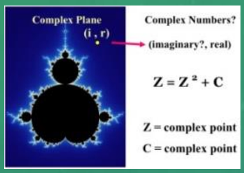
Split up into a positive and negative part

Split up the 28 further…

Simplify each part

This is usually written in this way

COMPLEX NUMBERS



$x^2 + 9 = 0$

You can use both real and imaginary numbers to solve equations

Solve the equation:

$x^2 + 9 = 0$

Subtract 9

$x^2 = -9$

Square root – we need to consider both positive and negative as we are solving an equation

$x = \pm\sqrt{-9}$

Split up

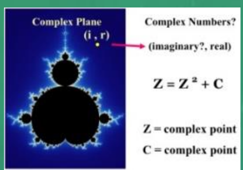
$x = \pm\sqrt{9}\sqrt{-1}$

Write in terms of i

$x = \pm 3i$

You should ensure you write full workings – once you have had a lot of practice you can do more in your head!

COMPLEX NUMBERS



You can use both real and imaginary numbers to solve equations

$x^2 + 6x + 25 = 0$

Solve the equation:

→ You can use one of two methods for this

→ Either 'Completing the square' or the Quadratic formula

$(x + 3)^2$

Imagine squaring the bracket

$(x + 3)(x + 3)$

This is the answer we get

$x^2 + 6x + 9$

The squared bracket gives us both the x^2 term and the 6x term

→ It only gives us a number of 9, whereas we need 25 – add 16 on!

Completing the square

$x^2 + 6x + 25 = 0$

Write a squared bracket, with the number inside being half the x-coefficient

$(x + 3)^2 + 16 = 0$

Subtract 16

$(x + 3)^2 = -16$

Square root

$x + 3 = \pm\sqrt{-16}$

Subtract 3

$x = -3 \pm \sqrt{-16}$

Split the root up

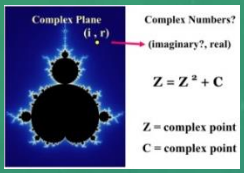
$x = -3 \pm \sqrt{16}\sqrt{-1}$

Simplify

$x = -3 \pm 4i$

If the x term is even, and there is only a single x^2, then completing the square will probably be the quickest method!

COMPLEX NUMBERS



You can use both real and imaginary numbers to solve equations

$x^2 + 6x + 25 = 0$

Solve the equation:

→ You can use one of two methods for this

a = 1

b = 6

c = 25

→ Either 'Completing the square' or the Quadratic formula

The Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sub in values

$$x = \frac{-6 \pm \sqrt{(6)^2 - (4 \times 1 \times 25)}}{2(1)}$$

Calculate the part under the root sign

$$x = \frac{-6 \pm \sqrt{-64}}{2}$$

Split it up

$$x = \frac{-6 \pm \sqrt{64}\sqrt{-1}}{2}$$

Simplify the roots

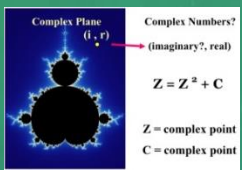
$$x = \frac{-6 \pm 8i}{2}$$

Divide all by 2

$$x = -3 \pm 4i$$

If the x^2 coefficient is greater than 1, or the x term is odd, the Quadratic formula will probably be the easiest method!

COMPLEX NUMBERS



You can use both real and imaginary numbers to solve equations

Simplify each of the following, giving your answers in the form:

$a + bi$

where: $a \in R$ and $b \in R$

This means a and b are real numbers

1) $(2+5i)+(7+3i)$

$= 9+8i$

Group terms together

2) $(2-5i)-(5-11i)$

$= 2-5i-5+11i$

$= -3+6i$

'Multiply out' the bracket

Group terms

3) $6(1+3i)$

$= 6+18i$

Multiply out the bracket

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra

$i \ \sqrt{-1}$

$i^2 = -1$

You can also use the following rule to simplify powers of i:

Multiply out the following bracket

$(2+3i)(4+5i)$

Multiply put like you would algebraically (eg) grid method, FOIL, smiley face etc)

$= 8 + 12i + 10i + 15i^2$

Group i terms, write i^2 as -1

$= 8 + 22i + 15(-1)$

Simplify

$= -7 + 22i$

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra

$i = \sqrt{-1}$

$i^2 = -1$

You can also use the following rule to simplify powers of i:

Express the following in the form a + bi

$(7 - 4i)^2$

Write as a double bracket

$= (7 - 4i)(7 - 4i)$

Multiply out

$= 49 - 28i - 28i + 16i^2$

Group i terms, write i^2 as -1

$= 49 - 56i + 16(-1)$

Simplify

$= 33 - 56i$

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra

$i = \sqrt{-1}$

$i^2 = -1$

You can also use the following rule to simplify powers of i:

Simplify the following:

$(2-3i)(4-5i)(1+3i)$

Start with the first 2 brackets

$(2-3i)(4-5i)$

Multiply out

$= 8 - 12i - 10i + 15i^2$

Group i terms, replace i^2 with -1

$= 8 - 22i + 15(-1)$

Simplify

$= -7 - 22i$

Now multiply this by the 3rd bracket

$(-7-22i)(1+3i)$

Multiply out the brackets

$= -7 - 22i - 21i - 66i^2$

Group i terms and replace i^2 with -1

$= -7 - 43i - 66(-1)$

Simplify

$= 59 - 43i$

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra

$i \; \sqrt{-1}$

$i^2 = -1$

You can also use the following rule to simplify powers of i:

Simplify:

1) i^3

$= i^2 \times i$ — Split up

$= -1 \times i$ — Replace i^2 with -1

$= -i$ — Simplify

2) i^4

$= i^2 \times i^2$ — Split up

$= -1 \times -1$ — Replace the i^2 terms with -1

$= 1$ — Simplify

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra

$i = \sqrt{-1}$

$i^2 = -1$

You can also use the following rule to simplify powers of i:

Simplify:

3) $(2i)^5$

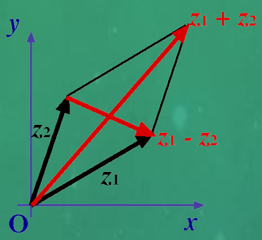
$= 2^5 \times i^5$ — Write both as a power of 5

$= 2^5 \times i^2 \times i^2 \times i$ — Split up the i terms

$= 32 \times -1 \times -1 \times i$ — Work out 2^5 and replace the i^2 terms

$= 32i$ — Simplify

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

You can write down the complex conjugate of a complex number, and it helps you divide one complex number by another

If a complex number is given by:

a + bi

Then the complex conjugate is:

a – bi

(You just reverse the sign of the imaginary part!)

Together, these are known as a **complex conjugate pair**

The complex conjugate of z is written as z*

Write down the complex conjugate of:

a) $2 + 3i$

$= 2 - 3i$

Reverse the sign of the imaginary term

b) $5 - 2i$

$= 5 + 2i$

Reverse the sign of the imaginary term

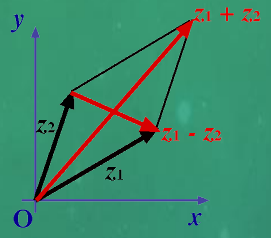
c) $1 - i\sqrt{5}$

$= 1 + i\sqrt{5}$

Reverse the sign of the imaginary term

1C

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

Find z + z*, and zz*, given that:

z = 2 - 7i

→ z* = 2 + 7i

$z+z^*$

$=(2-7i)+(2+7i)$

$=4$

Replace z and z*

Group terms

zz^*

$=(2-7i)(2+7i)$

$=4+14i-14i-49i^2$

$=4-49(-1)$

$=53$

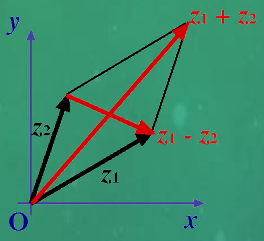
Replace z and z*

Multiply out

The i terms cancel out, replace i^2 with -1

Simplify

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

Find z + z*, and zz*, given that:

z = 2√2 + i√2

→ z* = 2√2 - i√2

$z + z^*$

$= (2\sqrt{2} + i\sqrt{2}) + (2\sqrt{2} - i\sqrt{2})$

$= 4\sqrt{2}$

Replace z and z*

Group terms

zz^*

$= (2\sqrt{2} + i\sqrt{2})(2\sqrt{2} - i\sqrt{2})$

$= 4\sqrt{4} + 2i\sqrt{4} - 2i\sqrt{4} - i^2\sqrt{4}$

$= 8 - (-1)(2)$

$= 10$

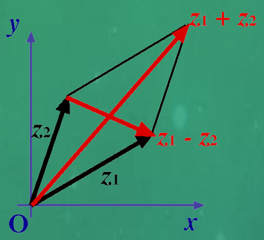
Replace z and z*

Multiply out

Some terms cancel out, replace i^2 with -1

Simplify

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

$(10+5i) \div (1+2i)$
Simplify:

With divisions you will need to write it as a fraction, then multiply both the numerator and denominator by the complex conjugate of the denominator

(This is effectively the same as rationalising when surds are involved!)

$$\frac{10+5i}{1+2i} \times \frac{1-2i}{1-2i}$$

Multiply by the complex conjugate of the denominator

$$= \frac{(10+5i)(1-2i)}{(1+2i)(1-2i)}$$

Expand both brackets

$$= \frac{10+5i-20i-10i^2}{1+2i-2i-4i^2}$$

Group i terms, replace the i^2 terms with -1 (use brackets to avoid mistakes)

$$= \frac{10-15i-10(-1)}{1-4(-1)}$$

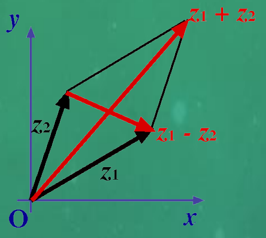
Simplify terms

$$= \frac{20-15i}{5}$$

Divide by 5

$$= 4-3i$$

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

$(5+4i) \div (2-3i)$

Simplify:

With divisions you will need to write it as a fraction, then multiply both the numerator and denominator by the complex conjugate of the denominator

(This is effectively the same as rationalising when surds are involved!)

$$\frac{5+4i}{2-3i} \times \frac{2+3i}{2+3i}$$

Multiply by the complex conjugate of the denominator

$$= \frac{(5+4i)(2+3i)}{(2-3i)(2+3i)}$$

Expand both brackets

$$= \frac{10+8i+15i+12i^2}{4+6i-6i-9i^2}$$

Group i terms, replace the i^2 terms with -1 (use brackets to avoid mistakes)

$$= \frac{10+23i+12(-1)}{4-9(-1)}$$

Simplify terms

$$= \frac{-2+23i}{13}$$

Split into two parts (this is useful for later topics!)

$$= -\frac{2}{13} + \frac{23}{13}i$$

Complex Number: A complex number is defined by $z = x + iy$, the symbol 'z' is called a complex variable where x, y are real and $i = \sqrt{-1}$. x is the real part and y is the imaginary part of z.

Modules and argument of z:

If $z = x + iy$ then the modules of z is, $|z| = \sqrt{x^2 + y^2}$ and argument of z is, $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

❒**Find the modules and argument of** $z = 1 - i$

Modules of z, $|z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

Now, Argument of z, $\arg(z) = \tan^{-1}\left(-\frac{1}{1}\right) = \tan(-1) = -45^\circ = 360^\circ - 45^\circ = 315^\circ$

❒**Find the modules and argument of** $z = \frac{1+i}{1-i}$

Given, $z = \frac{1+i}{1-i} = \frac{(1+i)}{(1-i)}\frac{(1+i)}{(1+i)} = \frac{(1+i)^2}{1^2 - i^2} = \frac{1+2i-1}{1+1} = \frac{2i}{2} = i$

Modules of z, $|z| = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

Argument of z, $\arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2} = 90°$

❒ **If $z = \frac{2+i}{2-i}$, find modules and argument of z**

Given that, $z = \frac{2+i}{2-i} = \frac{(2+i)}{(2-i)}\frac{(2+i)}{(2+i)} = \frac{(2+i)^2}{2^2-i^2} = \frac{4+2.2.i+i^2}{4+1} = \frac{3+4i}{5} = \frac{3}{5} + \frac{4}{5}i$

Modules of z , $|z| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9+16}{25}} = \sqrt{1} = 1$

Argument, $\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{\frac{4}{5}}{\frac{3}{5}}\right) = tan^{-1}\left(\frac{4}{3}\right) = 53.13°$

z $\arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2} = 90°$

Argument of ,

Exercise: Find the modules and argument of the following:

$(i)\quad z = 2+i \quad (ii)\quad z = 1-3i \quad (iii)\quad z = (1+i)^2$

$(iv)\quad z = \frac{2-i}{3+i} \quad (v)\quad z = \frac{3+i}{1+i}$

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

Making a direct substitution, we have

$$z = x + yi$$
$$z = (r\cos\theta) + i(r\sin\theta)$$
$$z = r(\cos\theta + i\sin\theta)$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

MODULUS, ARGUMENT AND COMPLEX CONJUGATE

$\bar{z} \coloneqq x - iy$ **complex conjugate**

$|z| := \sqrt{z\bar{z}}$ **modulus**

Therefore, $\frac{1}{z} = \frac{\bar{z}}{\bar{z}} \times \frac{1}{z} = \frac{\bar{z}}{|z|^2}$

$Re(z) := \frac{z+\bar{z}}{2}, Im(z) \coloneqq \frac{z-\bar{z}}{2i}$

$arg(z) \coloneqq arctan\, \frac{Re(z)}{Im(z)}$ **argument**

POLAR FORM

Euler Identity $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{ix} = 1 + ix + \frac{-1}{2!}x^2 + \frac{-i}{3!}x^3 + \cdots$$

$$\cos x = 1 + \frac{-1}{2!}x^2 + \cdots$$

$$i\sin x = ix + \frac{-i}{3!}x^3 + \cdots$$

Using identity, write $z = |z|e^{iarg(z)}$ (Polar form)

Find the polar form of $-4+4i$.

Solution

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$=\frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)$.

Week 11

Topics : Geometrical representations

Page no (157-159)

Examples: Describe geometrically the region of the following functions,

(i) z-3z+3=2 (ii) z+3+z-3=10

Solution: (i) Given that, $\left|\frac{z-3}{z+3}\right|=2$

$$\Rightarrow\frac{|z-3|}{|z+3|}=2 \quad \therefore\ |z-3|=2|z+3|$$

If, $z=x+iy$, then $|x+iy-3|=2|x+iy+3|$

$$\Rightarrow \left|(x-3)+iy\right| = 2\left|(x+3)+iy\right|$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x-3)^2+y^2}$$

Squaring both sides, we get,

$$(x-3)^2+y^2 = 4\left\{(x+3)^2+y^2\right\}$$

$$\Rightarrow x^2-6x+9+y^2 = 4\left(x^2+6x+9+y^2\right)$$

$$\Rightarrow x^2-6x+9+y^2 = 4x^2+24x+36+4y^2$$

$$\Rightarrow 3x^2+30x+3y^2+27=0$$

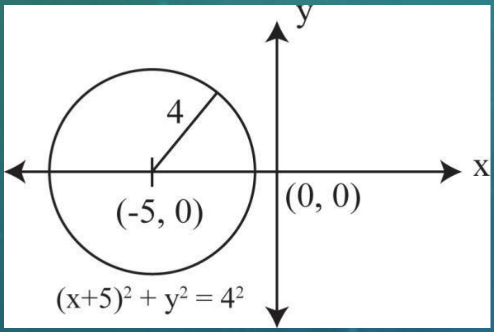
$$\Rightarrow x^2+10x+y^2+9=0$$

$$\Rightarrow x^2+2\cdot x\cdot 5+5^2+y^2 = 25-9$$

$$\Rightarrow (x+5)^2+y^2=16$$

$$\therefore\ (x+5)^2+y^2=4^2$$

which represents a circle of radius 4 and centre of (–5, 0)



(ii) Given that,

$$|z+3|+|z-3|=10$$

$$\Rightarrow |x+iy+3|+|x+iy-3|=10 \quad [\because\ z=x+iy]$$

$$\Rightarrow |(x+3)+iy|+|(x-3)+iy|=10$$

$$\Rightarrow \sqrt{(x+3)^2+y^2}+\sqrt{(x-3)^2+y^2}=10$$

$$\Rightarrow \sqrt{(x+3)^2+y^2}=10-\sqrt{(x-3)^2+y^2}$$

Squaring both sides we get,

$$(x+3)^2 + y^2 = 100 - 20\sqrt{(x-3)^2 + y^2} + (x-3)^2 + y^2$$

$$\Rightarrow x^2 + 6x + 9 + y^2 = 100 - 20\sqrt{(x-3)^2 + y^2} + x^2 - 6x + 9 + y^2$$

$$\Rightarrow 20\sqrt{(x-3)^2 + y^2} = 100 - 12x$$

$$\Rightarrow 5\sqrt{(x-3)^2 + y^2} = 25 - 3x$$

Again squaring both sides, we get,

$$\Rightarrow 25\left\{(x-3)^2 + y^2\right\} = (25-3x)^2$$

$$\Rightarrow 25\left(x^2 - 6x + 9 + y^2\right) = 625 - 150x + 9x^2$$

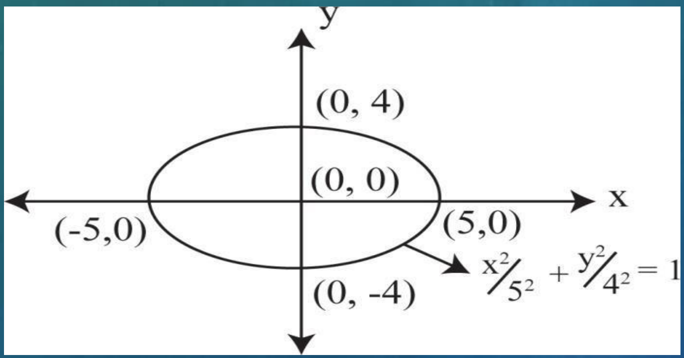
$$\Rightarrow 25x^2 - 150x + 225 + 25y^2 = 625 - 150x + 9x^2$$

$$\Rightarrow 16x^2 + 25y^2 = 400$$

$$\Rightarrow \frac{16}{400}x^2 + \frac{25}{400}y^2 = 1$$

$$\therefore \frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$$

which represents an ellipse passing through the points $(\pm 5, 0)$ and $(0, \pm 4)$



Exercise: Describe geometrically the region of the following functions

(i) $|z - i| = 2$ (ii) $|z + 2i| + |z - 2i| = 6$ (iii) $|z - 2| = 4|z + 2|$

(iv) $\left|\frac{z-1}{z+1}\right| = 6$ (v) $|z + 2| + |z - 2| = 3$

Week 12

Topics: Entire function, analytic function, Cauchy's Integral Formula

Page no (160-168)

Definition of Analytic Function:

A function f(z) is said to be analytic at a point z if z is an interior point of some region where f(z) is analytic. Hence the concept of analytic function at a point implies that the function is analytic in some circle with center at this point.

Definition of Entire Function:

1. Entire Functions are related to the field of complex analysis, which is also called Integral Function.
2. An entire function is a complex-valued function that is a complex differential in a neighborhood of each point in a domain in a complex coordinate space, also known as holomorphic on the whole complex plane.
3. Every entire function can be represented as a power series.

Examples of Entire Function:
Polynomials and Exponential Functions are the entire functions as they are holomorphic on the whole complex plane.

Cauchy's Integral Formula:

Let $f(z)$ is analytic inside and on the boundary of a simply connected closed curve C and a is any point inside C. The $f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \Rightarrow \oint_C \frac{f(z)}{z-a}dz=2\pi i\times f(a)$

Where C is traversed in the positive sense.

Question: Evaluate $\oint_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz$ Where C is the circle $|z|=3$

Solution: Let $\frac{1}{(z-1)(z-2)}=\frac{A}{z-2}+\frac{B}{z-1} \quad \Rightarrow 1=A(z-1)+B(z-2) \quad (1)$

Putting $z=2$ and $z=1$ in (1) we get $A=1$ and $B=-1$

Therefore $\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$

Hence $\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}dz-\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}dz \quad (2)$

Let $f(z)=\sin\pi z^2+\cos\pi z^2$. Since $z=1$ and $z=2$ are inside C and $f(z)$ is analytic inside and on C, so use com apply Cauchy's Integral formulae. Hence by Cauchy's integral formula we have $\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}dz=2\pi i\left\{\sin\pi(1)^2+\cos\pi(1)^2\right\}=2\pi i(\sin\pi+\cos\pi)=-2\pi i$

$$\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}dz=2\pi i\left\{\sin\pi(2)^2+\cos\pi(2)^2\right\}=2\pi i(\sin 4\pi+\cos 4\pi)=2\pi i$$

Substituting those values in (2), we get

$$\therefore \quad \oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=2\pi i+2\pi i=4\pi i$$

Example: Evaluate $\oint_c \frac{e^{-z}}{z-1}dz$, where C is the circle $|z|=2$

Solution: Here $f(z)=e^{-z}$ is an analytic function. So $f(z)$ is analytic inside and on the circle $|z|=2$. Also $z=1$. $\therefore |z|=1<2$. The point $z=1$ lies inside the circle $|z|=2$.

Hence by Cauchy integral formula we have $\oint_c \frac{e^{-z}}{z-1}dz=2\pi i\times f(1)=2\pi i\times e^{-1}=\frac{2\pi i}{e}$

Question: Show that $\oint_c \frac{e^{tz}}{z^2+1}dz = 2\pi i \sin t$, where C is the circle $|z|=3$ and $t>0$

Solution: Here $f(z)=e^{tz}$ is analytic inside and on the given circle $|z|=3$

Again $z^2+1 = z^2-(-1) = z^2 - i^2 = (z+i)(z-i)$

Let $\frac{1}{(z+i)(z-i)} = \frac{A}{z-i} + \frac{B}{z+i} \Rightarrow 1 = A(z+i)+B(z-i) \quad \ldots (1)$

Putting $z=i$ and $z=-i$ in (1) we get $A=\frac{1}{2i}$ and $B=-\frac{1}{2i}$

Therefore $\frac{1}{z^2+1} = \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{z+i}\right\}$

Therefore $\oint_c \frac{e^{tz}}{z^2+1}dz = \frac{1}{2i}\oint_c \frac{e^{tz}}{z-i}dz - \frac{1}{2i}\oint_c \frac{e^{tz}}{(z+i)}dz \quad \ldots (2)$

Hence by Cauchy's integral formula we get

$\oint_c \frac{e^{tz}}{z-i}dz = 2\pi i \times f(i) = 2\pi i e^{it}$ and $\oint_c \frac{e^{tz}}{z+i}dz = 2\pi i \times f(-i) = 2\pi i e^{-it}$

Substituting that value in (2), we get

$$\oint_c \frac{e^{tz}}{z^2+1}dz = \frac{1}{2i}\left(2\pi i\, e^{it}\right) - \frac{1}{2i}\left(2\pi i\, e^{-it}\right) = 2\pi i\left(\frac{e^{it}-e^{-it}}{2i}\right) = 2\pi i \,\text{sint}$$

❒ **Evaluate the following**:

$(i)\ \oint_C \frac{e^{3z}}{z-\pi i} \quad C:|z|=4 \qquad (ii)\ \oint_C \frac{\sin 3z}{z+\frac{\pi}{2}} \quad C:|z|=5 \qquad (iii)\ \oint_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} \quad C:|z|=3$

Cauchy-Riemann conditions

Analytic functions: If $f(z)$ is differentiable at $z = z_0$ and within the neighborhood of $z=z_0$, $f(z)$ is said to be **analytic** at $z = z_0$. A function that is analytic in the whole complex plane is called an *entire function*.

Cauchy-Riemann conditions for differentiability

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann conditions

Conversely, if the Cauchy-Riemann conditions are satisfied, $f(z)$ is differentiable:

$$\frac{df}{dz}=\lim_{\Delta z\to 0}\frac{\Delta f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x+i\Delta y}=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y}{\Delta x+i\Delta y}$$

$$=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)}{\Delta x+i\Delta y}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\quad \text{and}=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right).$$

More about Cauchy-Riemann conditions:

1) It is a very strong restraint to functions of a complex variable.

2) $\frac{df}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial(iy)}+i\frac{\partial v}{\partial(iy)}.$

3) $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=0\Rightarrow \nabla u\cdot\nabla v=0\Rightarrow \nabla u\perp\nabla v\Rightarrow u=c_1\perp v=c_2$

4) Equivalent to $\frac{\partial f}{\partial z^*}=0$, so that $f(z,z^*)$ only depends on z:

$$\frac{\partial f}{\partial z^*}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}=\frac{\partial f}{\partial x}\frac{1}{2}+\frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)=0\Rightarrow\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=0\Rightarrow\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)=0\Rightarrow\cdots$$

e.g., $f=x-iy$ is everywhere continuous but not analytic.

Cauchy's theorem

Cauchy's integral theorem

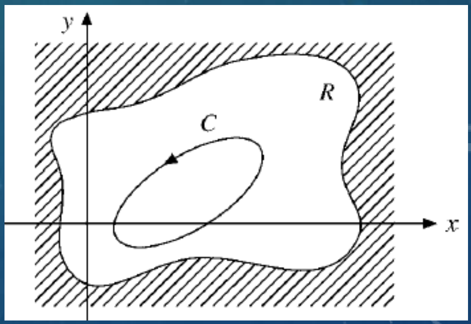
Contour integral:

$$\int_{z_1}^{z_2} f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx+udy)$$

Cauchy's integral theorem: If $f(z)$ is analytic in a simply connected region R, [and $f'(z)$ is continuous throughout this region,] then for any closed path C in R, the contour integral of $f(z)$ around C is zero: $\oint_C f(z)dz = 0$

Proof using Stokes' theorem: $\oint_C \mathbf{V}\cdot d\boldsymbol{\lambda} = \iint_S \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}$

$$\oint_C \left(V_x dx + V_y dy\right) = \iint_S \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy$$



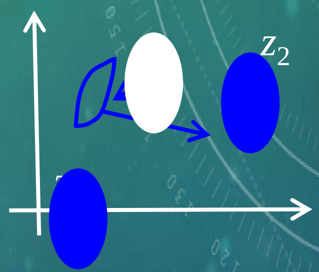
$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx+udy)$$

$$= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

$$= 0$$

Cauchy-Goursat proof: The continuity of $f'(z)$ is not necessary.

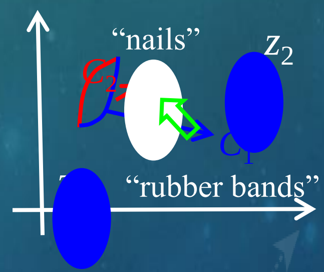
Corollary: An open contour integral for an analytic function is independent of the path, if there is no singular points between the paths.



$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)dz$$

Contour deformation theorem:

A contour of a complex integral can be arbitrarily deformed through an analytic region without changing the integral.



1) It applies to both open and closed contours.
2) One can even split closed contours.
Proof: Deform the contour bit by bit.

Examples:
1) Cauchy's integral theorem.
(Let the contour shrink to a point.)
2) Cauchy's integral formula.
(Let the contour shrink to a small circle.)

Cauchy’s integral formula

Cauchy’s integral formula:

If $f(z)$ is analytic within and on a closed contour C, then for any point z_0 within C,

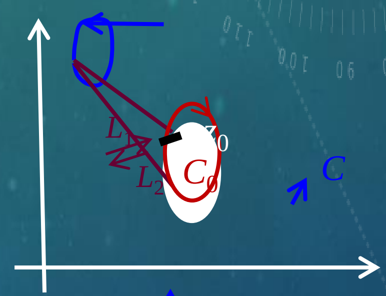
$$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

Proof:

$$\oint_C \frac{f(z)}{z-z_0}dz+\oint_{L_1} \frac{f(z)}{z-z_0}dz+\oint_{C_0} \frac{f(z)}{z-z_0}dz+\oint_{L_2} \frac{f(z)}{z-z_0}dz=0$$

$$\oint_C \frac{f(z)}{z-z_0}dz=-\oint_{C_0} \frac{f(z)}{z-z_0}dz=-\int_{2\pi}^{0} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}rie^{i\theta}d\theta \quad (\text{Let } r\to 0)$$

$$=2\pi i f(z_0)$$



Can directly use the contour deformation theorem.

Week 13 & 14
Topics: Harmonic and Conjugate harmonic
Page no (168-173)

Harmonic Function: Harmonic functions occur regularly and play an essential role in maths and other domains like physics and engineering. In complex analysis, harmonic functions are called the solutions of the Laplace equation. Every harmonic function is the real part of a holomorphic function in an associated domain. In this article, you will learn the definition of harmonic function, along with some fundamental properties.

Before learning about harmonic functions, let's recall the definition of the Laplace equation.

An equation having the second-order partial derivatives of the form.

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U. This is usually written as

$$\nabla^2 f = 0$$

or

$$\Delta f = 0$$

Question: Show that is a harmonic function and hence finds its harmonic conjugate if is analytic.

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

v $\quad f(z) = u + iv$

Solution: Given that

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1\right)$$

$$= 3x^2 - 3y^2 + 6x = \phi_1(x, y), say \qquad (1)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1\right)$$
$$= -6xy - 6y = \phi_2(x, y), say \qquad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 - 3y^2 + 6x\right) = 6x + 6 \qquad (3)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-6xy - 6y\right) = -6x - 6 \qquad (4)$$

Adding (3) and (4) we get, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0 \quad \Rightarrow \nabla^2 u = 0$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z,\ y = 0$ in (1) and (2), we get $\phi_1(z, 0) = 3z^2 + 6z$ and $\phi_2(z, 0) = 0$

By Milne's method we have $f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0) = 3z^2 + 6z$

So $f(z) = \int \left(3z^2 + 6z\right)dz = z^3 + 3z^2 + c_1 + ic_2$, where $c_1 + ic_2$ is a complex constant.

Then

$$u + iv = (x + iy)^3 + 3(x + iy)^2 + c_1 + ic_2$$
$$u + iv = x^3 + 3x^2 iy + 3x i^2 y^2 + i^3 y^3 + 3\left(x^2 + 2xiy + i^2 y^2\right) + c_1 + ic_2$$
$$u + iv = x^3 + 3x^2 iy - 3xy^2 - iy^3 + 3x^2 + 6ixy - 3y^2 + c_1 + ic_2$$
$$u + iv = x^3 - 3xy^2 + 3x^2 - 3y^2 + c_1 + i\left(3x^2 y - y^3 + 6xy + c_2\right)$$

Equating imaginary part from both sides we get

$$v = 3x^2 y - y^3 + 6xy + c_2$$

<u>Question:</u> Show that $u = 3x^2 y + 2x^2 - y^3 - 2y^2$ is a harmonic function and hence finds its harmonic conjugate v if $f(z) = u + iv$ is analytic.

Solution: Given that $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y + 2x^2 - y^3 - 2y^2\right)$$
$$= 6xy + 4x = \phi_1(x,y), say \quad \cdots\cdots \quad (1)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y + 2x^2 - y^3 - 2y^2\right)$$
$$= 3x^2 - 3y^2 - 4y = \phi_2(x,y), say \quad \cdots\cdots \quad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(6xy + 4x\right) = 6y + 4 \quad \cdots\cdots \quad (3)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(3x^2 - 3y^2 - 4y\right) = -6y - 4 \quad \cdots\cdots \quad (4)$$

Adding (3) and (4) we get, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6x - 4 = 0 \quad \Rightarrow \nabla^2 u = 0$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z, y = 0$ in (1) and (2), we get $\phi_1(z,0) = 4z$ and $\phi_2(z,0) = 3z^2$

By Milne's method we have $f'(z) = \phi_1(z,0) - i\,\phi_2(z,0) = 4z - i3z^2$

So $f(z) = \int\left(4z - i3z^2\right)dz = 2z^2 - iz^3 + c_1 + ic_2$, where $c_1 + ic_2$ is a complex constant.

Then

$$u + iv = 2(x+iy)^2 - i(x+iy)^3 + c_1 + ic_2$$
$$u + iv = 2\left(x^2 + 2xiy + i^2y^2\right) - i\left(x^3 + 3x^2iy + 3x\,i^2y^2 + i^3y^3\right) + c_1 + ic_2$$
$$u + iv = 2x^2 + 4xiy - 2y^2 - ix^3 + 3x^2y + i3x\,y^2 - y^3 + c_1 + ic_2$$
$$u + iv = 3x^2y + 2x^2 - y^3 - 2y^2 + i\left(4xy - x^3 + 3xy^2 + c_2\right)$$

Equating imaginary part from both sides we get

$$v = 4xy - x^3 + 3xy^2 + c_2$$

Week 14

Topics: Finding conjugate harmonics

Page no (40-43)

Question: Show that $u = x^3 + 6x^2y - 3xy^2 - 2y^3$ is a harmonic function and hence finds its harmonic conjugate v if $f(z) = u + iv$ is analytic.

Solution: Given that $u = x^3 + 6x^2y - 3xy^2 - 2y^3$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + 6x^2y - 3xy^2 - 2y^3\right)$$
$$= 3x^2 + 12xy - 3y^2 = \phi_1(x, y), say \quad (1)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + 6x^2y - 3xy^2 - 2y^3\right)$$
$$= 6x^2 - 6xy - 6y^2 = \phi_2(x, y), say \quad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 + 12xy - 3y^2\right) = 6x + 12y \quad (3)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(6x^2 - 6xy - 6y^2\right) = -6x - 12y \quad (4)$$

Adding (3) and (4) we get, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y = 0 \quad \Rightarrow \nabla^2 u = 0$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z, y = 0$ in (1) and (2), we get $\phi_1(z, 0) = 3z^2$ and $\phi_2(z, 0) = 6z^2$

By Milne's method we have $f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0) = 3z^2 - i6z^2 = 3(1 - 2i)z^2$

So $f(z) = \int 3(1 - 2i)z^2\, dz = (1 - 2i)z^3 + c_1 + ic_2$, where $c_1 + ic_2$ is a complex constant.

Then

$$u+iv=(1-2i)(x+iy)^3+c_1+ic_2$$
$$u+iv=\left(x^3+3x^2iy+3x\,i^2y^2+i^3y^3\right)-2i\left(x^3+3x^2iy+3x\,i^2y^2+i^3y^3\right)+c_1+ic_2$$
$$u+iv=x^3+3x^2iy-3xy^2-iy^3-2ix^3-6x^2i^2y-6x\,i^3y^2-2i^4y^3+c_1+ic_2$$
$$u+iv=x^3+3x^2iy-3xy^2-iy^3-2ix^3+6x^2y+6i\,xy^2-2y^3+c_1+ic_2$$
$$u+iv=x^3+6x^2y-3xy^2-2y^3+c_1+i\left(3x^2y-2x^3-y^3+6xy^2+c_2\right)$$

Equating imaginary part from both sides we get $v=3x^2y-2x^3-y^3+6xy^2+c_2$

<u>Question:</u> Show that $u=e^x(x\cos y-y\sin y)$ is a harmonic function and hence finds its harmonic conjugate v if $f(z)=u+iv$ is analytic.

<u>Solution:</u> Given that $u=e^x(x\cos y-y\sin y)$

$$\therefore\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left[e^x(x\cos y-y\sin y)\right]=e^x(x\cos y-y\sin y)+e^x\cos y=\phi_1(x,y),say\quad \cdots\cdots\quad(1)$$

$$\therefore\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[e^x(x\cos y-y\sin y)\right]=e^x(-x\sin y-\sin y-y\cos y)=\phi_2(x,y),say\quad \cdots\cdots\quad(2)$$

$$\therefore\frac{\partial^2 u}{\partial x^2}=e^x(x\cos y-y\sin y)+e^x\cos y+e^x\cos y$$
$$=e^x(x\cos y-y\sin y+2\cos y)\quad \cdots\cdots\quad(3)$$

$$\therefore\frac{\partial^2 u}{\partial y^2}=e^x(-x\cos y-\cos y-\cos y+y\sin y)\quad \cdots\cdots\quad(4)$$

Adding (3) and (4) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^x(x\cos y-y\sin y+2\cos y)+e^x(-x\cos y-\cos y-\cos y+y\sin y)$$
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^x(x\cos y-y\sin y+2\cos y-x\cos y-\cos y-\cos y+y\sin y)=e^x\times 0=0$$
$$\Rightarrow\nabla^2u=0$$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z)=u+iv$ is analytic.

Putting $x=z, y=0$ in (1) and (2), we get $\phi_1(z,0)=e^z(z\cdot 1-0)+e^z\cdot 1=ze^z+e^z$ & $\phi_2(z,0)=0$

By Milne's method we have $f'(z)=\phi_1(z,0)-i\,\phi_2(z,0)=ze^z+e^z$

So $f(z)=\int(ze^z+e^z)dz=ze^z-\int e^z dz+e^z=ze^z+c_1+ic_2$, c_1+ic_2 is a complex constant.
Then

$$u+iv=(x+iy)e^{x+iy}+c_1+ic_2$$
$$u+iv=(x+iy)e^x\cdot e^{iy}+c_1+ic_2$$
$$u+iv=(x+iy)e^x(\cos y+i\sin y)+c_1+ic_2$$
$$u+iv=e^x(x\cos y+ix\sin y+iy\cos y-y\sin y)+c_1+ic_2$$
$$u+iv=e^x(x\cos y-y\sin y)+ie^x(x\sin y+y\cos y)$$

Equating imaginary part from both sides we get $v=e^x(x\sin y+y\cos y)+c_2$

Week 15:

Topics: Cauchy's Residue

Page no (173-176)

Question: Show that $\oint_c \frac{e^{tz}}{(z^2+1)^2}dz$, where C is the circle $|z|=3$ and $t>0$

Solution: Here the circle $|z|=3$. The poles of $\frac{e^{tz}}{(z^2+1)^2}$ are obtain by solving the equation
$(z^2+1)^2=0 \quad \Rightarrow z^2+1=0 \quad \Rightarrow z^2-(-1)=0 \quad \Rightarrow z^2-i^2=0 \quad \Rightarrow (z+i)(z-i)=0 \quad \therefore z=-i, i$

Both poles are double poles and lie inside the circle C, since $|i|=|-i|=1<3$ and $|i|=1<3$

Residue at $z=i$ is

$$\lim_{z\to i}\frac{1}{1!}\frac{d}{dz}\left[(z-i)^2\frac{e^{tz}}{(z-i)^2(z+i)^2}\right]=\lim_{z\to i}\frac{d}{dz}\left[\frac{e^{tz}}{(z+i)^2}\right]$$

$$=\lim_{z\to i}\frac{(z+i)^2\times te^{tz}-e^{tz}\times 2(z+i)}{(z+i)^4}$$

$$=\lim_{z\to i}\frac{t(z+i)e^{tz}-2e^{tz}}{(z+i)^3}=\frac{t(i+i)e^{it}-2e^{it}}{(i+i)^3}$$

$$=\frac{2it\,e^{it}-2e^{it}}{8i^3}=\frac{2(it-1)e^{it}}{-8i}$$

$$=\frac{(it-1)e^{it}}{-4i}=\frac{-(t+i)e^{it}}{4}$$

Similarly, residue at $z=-i$ is $\dfrac{-(t-i)e^{-it}}{4}$

Therefore, by Cauchy's residue theorem we have

$$\oint_c\frac{e^{tz}}{(z^2+1)^2}dz=2\pi i[sum\ \ of\ \ residue]=-2\pi i\left[\frac{(t+i)e^{it}+(t-i)e^{-it}}{4}\right]$$

$$=-2\pi i\left[\frac{t(e^{it}+e^{-it})+i(e^{it}-e^{-it})}{4}\right]$$

$$=-2\pi i\left[\frac{2t\cos t+i\times 2i\sin t}{4}\right]$$

$$=-\pi i\left[\frac{4t\cos t+4i^2\sin t}{4}\right]$$

$$=-\pi i(t\cos t-\sin t)=\pi i(\sin t-t\cos t)$$

Question: Show that $\displaystyle\oint_c\frac{e^{-iz}}{(z+3)(z-i)^2}dz$, where C is the circle $|z-1|=2$

Solution: Here the circle $|z|=3$. The poles of $\dfrac{e^{-iz}}{(z+3)(z-i)^2}$ are obtain by solving the equation

$$(z+3)(z-i)^2=0\quad\Rightarrow z+3=0\quad and\quad (z-i)^2=0\quad\Rightarrow z=-3\quad and\quad z=i,i$$

Since $|i| = 1 < 2$ and $|-3| = 3 > 2$. Thus the pole $z = i$ is double pole and lie inside the circle C.

Residue at $z = i$ is

$$\lim_{z \to i} \frac{1}{1!} \frac{d}{dz}\left[(z-i)^2 \frac{e^{-iz}}{(z+3)(z-i)^2} \right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\frac{e^{-iz}}{(z+3)} \right]$$

$$= \lim_{z \to i} \frac{(z+3)\times(-ie^{-iz}) - e^{-iz}\times 1}{(z+3)^2}$$

$$= \frac{(i+3)(-ie) - e}{(i+3)^2} = \frac{(1-3i-1)e}{-1+6i+9}$$

$$= \frac{-3ie}{8+6i} = \frac{-3ie(4-3i)}{2(4+3i)(4-3i)}$$

$$= \frac{-12ie + 9i^2 e}{2(16+9)}$$

$$= \frac{-12ie - 9e}{50}$$

Therefore, by Cauchy's residue theorem we have

$$\oint_c \frac{e^{-iz}}{(z+3)(z-i)^2} dz = 2\pi i [residue \ at \ z = i]$$

$$= 2\pi i \times \left(\frac{-12ie - 9e}{50} \right)$$

$$= \frac{-12i^2 \pi e - 9i\pi e}{50}$$

$$= \frac{(12-9i)\pi e}{25}$$

Week 16
Topics: Laurent series
Page no (176-183)

Laurent expansion

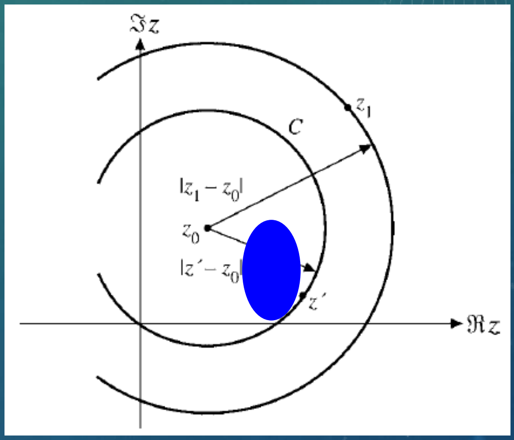
Taylor expansion for functions of a complex variable:
Expanding an analytic function $f(z)$ about $z = z_0$, where z_1 is the nearest singular point.

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz' = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)-(z-z_0)}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}dz' = \frac{1}{2\pi i}\oint_C \frac{\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n f(z')}{(z'-z_0)}dz'$$

$$= \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty}\frac{(z-z_0)^n f(z')}{(z'-z_0)^{n+1}}dz' = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$



Laurent expansion:

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n,\quad a_n=\frac{1}{2\pi i}\oint_C\frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

1) Singular points of the integrand.
   For $n < 0$, the singular points are determined by $f(z)$. For $n \geq 0$, the singular points are determined by both $f(z)$ and $1/(z'-z_0)^{n+1}$.
2) If $f(z)$ is *analytic* inside C, then the Laurent series reduces to a Taylor series:

$$a_n=\begin{cases}\dfrac{f^{(n)}(z_0)}{n!}, & n\geq 0,\\ 0, & n<0.\end{cases}$$

3) Although a_n has a general contour integral form, In most times we need to use straight forward complex algebra to find a_n.

Laurent expansion: Examples

Example 1: Expand $f(z)=\frac{z^3}{(z-1)^2}$ about $z_0=1$.

$$\frac{z^3}{(z-1)^2}=\frac{[(z-1)+1]^3}{(z-1)^2}=\frac{(z-1)^3+3(z-1)^2+3(z-1)+1}{(z-1)^2}=\frac{1}{(z-1)^2}+\frac{3}{z-1}+3+(z-1)$$

Example 2: Expand $f(z)=\frac{1}{z^2+1}$ about $z_0=i$.

$$f(z)=\frac{1}{z^2+1}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{2i+z-i}\right)$$

$$=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{2i}\cdot\frac{1}{1+\frac{z-i}{2i}}\right)=\frac{1}{2i}\frac{1}{z-i}-\frac{1}{(2i)^2}\sum_{n=0}^{\infty}\left(-\frac{1}{2i}\right)^n(z-i)^n$$

$$=-\frac{i}{2}\frac{1}{z-i}+\frac{1}{4}+\frac{i}{8}(z-i)+\cdots$$

Branch points and branch cuts

Singularities

Poles: In a Laurent expansion $f(z)=\sum_{m=-\infty}^{\infty} a_m (z-z_0)^m$, if $a_m=0$ for $m<-n<0$ and $a_{-n}\neq 0$,

then z_0 is said to be *a pole of order n*.

A pole of order 1 is called a *simple pole*.

A pole of infinite order (when expanded about z_0) is called an *essential singularity*.

The behavior of a function $f(z)$ at infinity is defined using the behavior of $f(1/t)$ at $t = 0$.

Examples:

$$1)\ \frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=\frac{1}{2i}\left[-\frac{1}{z+i}-\frac{1}{2i-(z+i)}\right]=-\frac{1}{2i}\frac{1}{z+i}+\frac{1}{4}\frac{1}{1-(z+i)/2i}$$

$$=-\frac{1}{2i}\frac{1}{z+i}+\frac{1}{4}\left[1+\frac{z+i}{2i}+\left(\frac{z+i}{2i}\right)^2+\cdots\right] \text{ has a single pole at } z=-i.$$

$$2)\ \sin z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!},\quad \sin\frac{1}{t}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{t^{2n+1}}$$

sinz thus has an essential singularity at infinity.

3) z^2+1 has a pole of order 2 at infinity.

2. $f(z)=\sqrt{(z-1)(z+1)}$

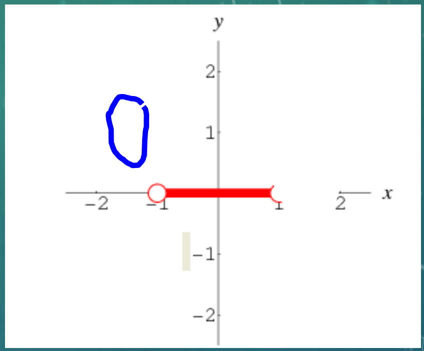
We can choose a branch cut from z = -1 to z = 1 (or any curve connecting these two points). The function will be single-valued, because both points will be circled.

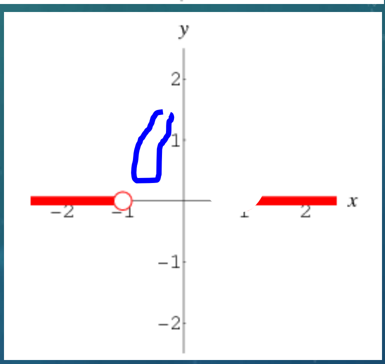
Alternatively, we can choose a branch cut which connects each branch point to infinity. The function will be single-valued, because neither points will be circled.

It is notable that these two choices result in different functions. E.g., if $f(i)=\sqrt{2}i$, then

$f(-i)=-\sqrt{2}i$ for the first choice and $f(-i)=\sqrt{2}i$

for the second choice.

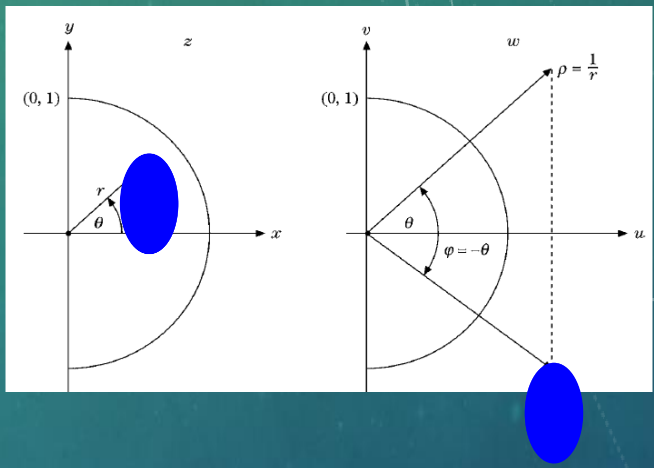




Inversion:

$$w = \frac{1}{z}, \text{ or}$$

$$\rho e^{i\varphi} = \frac{1}{re^{i\theta}} \Rightarrow \begin{cases} \rho = \dfrac{1}{r} \\ \varphi = -\theta \end{cases}$$



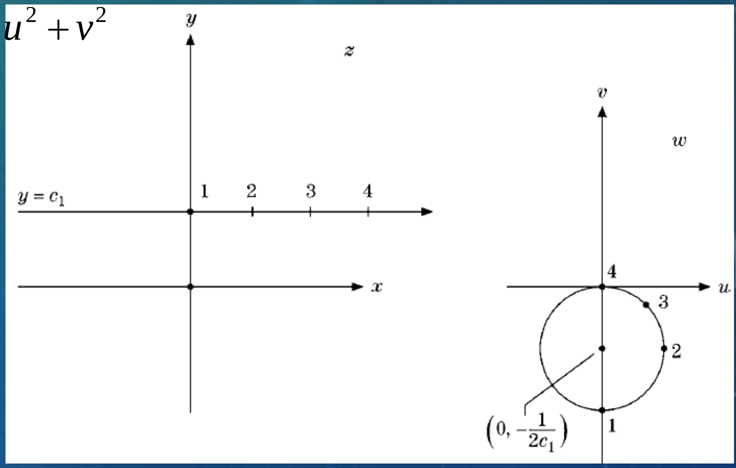
In Cartesian coordinates:

$$w = \frac{1}{z} \Rightarrow u + iv = \frac{1}{x+iy} \Rightarrow \begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = -\dfrac{y}{x^2+y^2} \end{cases}, \begin{cases} x = \dfrac{u}{u^2+v^2} \\ y = -\dfrac{v}{u^2+v^2} \end{cases}.$$

A straight line is mapped into a circle:

$$y = ax + b \Rightarrow -\frac{v}{u^2+v^2} = \frac{au}{u^2+v^2} + b$$

$$\Rightarrow b(u^2+v^2) + au + v = 0.$$



Question: Expand the function $f(z)=\dfrac{1}{(z+1)(z+3)}$ in a **Laurent** series for the following region: $(i)\quad 1<|z|<3 \qquad (ii)\quad |z|<1$

Solution: Given
$$f(z)=\frac{1}{(z+1)(z+3)}=\frac{1}{(z+1)(-1+3)}+\frac{1}{(-3+1)(z+3)}=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$

$(i)\quad 1<|z|<3 \Rightarrow 1<|z| \quad and \quad |z|<3 \quad \Rightarrow \dfrac{1}{|z|}<1 \quad and \quad \dfrac{|z|}{3}<1$

We write $f(z)$ in a manner so that the binomial expansion is valid for

$1<|z|<3 \quad \Rightarrow \dfrac{1}{|z|}\ and\ \dfrac{|z|}{3}<1$

$$f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$
$$=\frac{1}{2z}\times\frac{1}{1+\frac{1}{z}}-\frac{1}{6}\times\frac{1}{1+\frac{z}{3}}$$
$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$
$$=\frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)-\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$
$$=\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots\right)-\left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\cdots\right)$$

$(i)\quad |z|<1 \Rightarrow |z|<1 \quad and \quad |z|<3 \quad \Rightarrow |z|<1 \quad and \quad \dfrac{|z|}{3}<1$

We write $f(z)$ in a manner so that the binomial expansion is valid for $|z|<1 \quad and \quad \dfrac{|z|}{3}<1$

$$f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$

$$=\frac{1}{2}(1+z)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2}\left(1-z+z^2-z^3+\cdots\right)-\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$

$$=\left(\frac{1}{2}-\frac{1}{6}\right)+\left(-\frac{1}{2}+\frac{1}{18}\right)z+\left(\frac{1}{2}-\frac{1}{54}\right)z^2+\left(-\frac{1}{2}+\frac{1}{162}\right)z^3+\cdots$$

$$=\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\frac{40}{81}z^3+\cdots$$

<u>Question:</u> Expand the function $f(z)=\dfrac{z^2+1}{(z+1)(z-2)}$ in a **Laurent** series for the following region: $(i)\quad 1<|z|<2 \qquad (ii)\quad 0<|z|<1$

<u>Solution:</u> Given $f(z)=\dfrac{z^2+1}{(z+1)(z-2)}$. Let $\dfrac{z^2+1}{(z+1)(z-2)}=1+\dfrac{a}{z+1}+\dfrac{b}{z-2}$

Therefore, $z^2+1=(z+1)(z-2)+a(z-2)+b(z+1)$

When $z=-1$, then $2=0+a(-3)+0 \quad \Rightarrow a=\dfrac{-2}{3}$. When $z=2$, then $5=0+0+3b \quad \Rightarrow b=\dfrac{5}{3}$

$$\therefore \quad f(z)=\frac{z^2+1}{(z+1)(z-2)}=1-\frac{2/3}{z+1}+\frac{5/3}{z-2} \quad \cdots \quad (1)$$

(i) $1<|z|<3 \Rightarrow 1<|z| \quad and \quad |z|<2 \quad \Rightarrow \dfrac{1}{|z|}<1 \quad and \quad \dfrac{|z|}{2}<1$

Therefore, from (1) we have

$$f(z)=1-\frac{2/3}{z\left(1+\frac{1}{z}\right)}+\frac{5/3}{-2\left(1-\frac{z}{2}\right)}$$

$$=1-\frac{2}{3z}\left(1+\frac{1}{z}\right)^{-1}-\frac{5}{6}\left(1-\frac{z}{2}\right)^{-1}$$

$$=1-\frac{2}{3z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)-\frac{5}{6}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$=\cdots+\frac{2}{3z^4}-\frac{2}{3z^3}+\frac{2}{3z^2}-\frac{2}{3z}+\frac{1}{6}-\frac{5z}{12}-\frac{5z^2}{24}-\frac{5z^3}{48}-\cdots$$

$$(i)\quad 0<|z|<1 \Rightarrow |z|<1. \quad Also \quad |z|<1 \quad \Rightarrow |z|<2 \quad \Rightarrow \quad \frac{|z|}{2}<1$$

Therefore, from (1) we have

$$f(z)=1-\frac{2/3}{(z+1)}+\frac{5/3}{-2\left(1-\frac{z}{2}\right)}$$

$$=1-\frac{2}{3}(1+z)^{-1}-\frac{5}{6}\left(1-\frac{z}{2}\right)^{-1}$$

$$=1-\frac{2}{3}\left(1-z+z^2-z^3+\cdots\right)-\frac{5}{6}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$=\left(1-\frac{1}{2}-\frac{5}{6}\right)+\left(\frac{2}{3}-\frac{5}{12}\right)z+\left(-\frac{2}{3}-\frac{5}{24}\right)z^2+\left(\frac{2}{3}-\frac{5}{48}\right)z^3+\cdots$$

$$=-\frac{1}{2}+\frac{1}{4}z-\frac{7}{8}z^2+\frac{9}{16}z^3+\cdots$$

Week 17

Topics: Applications

Page no (184-187)

Question: Solve the Partial Differential Equation

$$x\left(y^2-z^2\right)p+y^2(z-x)q=z\left(x^2-y^2\right)$$

Solution: Given the partial differential equation is

$$x\left(y^2-z^2\right)p+y^2(z-x)q=z\left(x^2-y^2\right)$$

The Lagrange auxiliary equations of (1) are

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)} \quad \cdots \quad (2)$$

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)}=\frac{xdx+ydy+zdz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-y^2z^2}=\frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c}{2}$$
$$\therefore\ x^2+y^2+z^2=c \quad \cdots \cdots \quad (3)$$

Again (2), implies

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$
$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrating, we get

$$\ln x+\ln y+\ln z=\ln c_1$$
$$\Rightarrow \ln xyz=\ln c_1$$
$$\therefore\ xyz=c_1 \quad \cdots \cdots \quad (4)$$

From (3) and (4), the required general solution is $x^2+y^2+z^2=\phi\left(xyz\right)$, where ϕ being an arbitrary function.

$$\left[\quad p=\frac{\partial}{\partial x},\quad q=\frac{\partial}{\partial y}\quad\right]$$

Question: Find the particular integrals of the following partial differential equation to represent surface passing through the given lines.

$$x\left(y^2+z\right)p-y\left(x^2+z\right)q=z\left(x^2-y^2\right);\quad x=t,\quad y=-t,\quad z=1$$

Solution: Given the partial differential equation is

$$x\left(y^2+z\right)p-y\left(x^2+z\right)q=z\left(x^2-y^2\right) \quad \cdots \cdots \quad (1)$$

And the line is $x=t,\quad y=-t,\quad z=1 \quad \cdots \cdots \quad (2)$

The Lagrange auxiliary equations of (1) are

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{z\left(x^2-y^2\right)} \quad \cdots (3)$$

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{z\left(x^2-y^2\right)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}d=0$$

Integrating, we get

$$\ln x+\ln y+\ln z=\ln c_1$$

$$\Rightarrow \ln xyz=\ln c_1$$

$$\therefore\ xyz=c_1 \quad \cdots \quad (4)$$

Again (3), implies

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{z\left(x^2-y^2\right)}=\frac{xdx+ydy-dz}{x^2y^2+x^2z-y^2x^2-y^2z-x^2z+y^2z}=\frac{xdx+ydy-dz}{0}$$

$$\Rightarrow xdx+ydy-dz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}-z=\frac{c_2}{2}$$

$$\therefore\ x^2+y^2-z=c_2 \quad \cdots (5)$$

By (2) and (4), $-t^2=c_1$ and (5) implies $2t^2-2=c_2 \quad \Rightarrow 2c_1+c_2+2=0 \quad \cdots \quad (6)$

Now, putting the values of c_1 and c_2 from (4) and (5) in (6), we obtain

$2xyz+x^2+y^2-2z+2=0$ which is the required solution.