University of Global Village (UGV)

Course Title: Fourier Analysis, Laplace Transform and Linear Algebra
Course Code: 0541-2101
Credit: 3
CIE: 90 marks
SEE: 60 Marks
Exam hour: 3

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Course Outlines

Course Code: Math 0541-2101 Semester End Exam (SEE): 3 Hours	Credit: 03 CIE Marks: 90 SEE marks: 60
Course Learning Outcomes (CLO): After successful completion of the course students will be able to -	
CLO1 Define the basic terminology and theorems associated with Fourier Analysis, Laplace Transformation and Linear algebra.	
CLO2 Properties of Laplace and Inverse Laplace Transformation and Laplace Transformation of derivatives, and Applications.	
CLO3 Describe Fourier Series, Fourier Sine and Cosine Series, Orthogonal Functions, Fourier Integrals.	
CLO4 Matrices, Operations, Type of matrices, related theorems, and Different applications.	
CLO5 Apply the acquired concepts of Fourier Analysis, Linear algebra, and Laplace Transformation in engineering.	

Course Content Summary

SL.	Content of Courses	Hrs	CLO's
1	Definition of Laplace transformations, Some important properties of Laplace Transformations, and some related mathematics, Laplace transformations of some elementary functions, Laplace Transformation of 1st and 2nd derivatives and general term derivatives, multiplication by t power n and division by t, Inverse Laplace transformations, Inverse Laplace transformations of some elementary functions, Inverse Laplace Transformation of 1st and 2nd derivatives and, Ordinary Differential	12	CLO1, CLO2

	Equations with Constant Coefficients, Related Mathematics. Applications to electrical circuits, L-R circuit related Problems.		
2	Definition of Fourier series, Periodic Function, Even and Odd Function, Piecewise Continuous Functions, Dirichlet Conditions, Parsivals Identity, Fourier Series, Some important properties of Fourier series, Half range Fourier Sine or Cosine Series, and Related mathematics, Convergence of Fourier series, Definition of orthogonal Functions, Orthogonality, Orthogonal series	6	CLO1, CLO3
3	Application of Fourier series in engineering, Boundary value Problem, Laplace equation and Related mathematics, Fourier integrals, Fourier Transforms, Fourier sine and cosine Transforms, Convolution Theorem, Application of Fourier integrals, Related mathematics.	6	CLO3, CLO5
4	Matrix, its operations and classifications, related theorems, Determinant and Inverse Matrix, System of Linear Equations, Matrix method for solving linear system, Cramer's rule for solving linear systems and solving related problems, Rank of a matrix and eigenvalue, Cayley-Hamilton theorem, Application of Linear Algebra in Engineering and related mathematics.	10	CLO4, CLO1

Course Plan Specifying content, CLO's, Teaching Learning, and Assessment strategy mapping with CLO's

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
1	Laplace transformation • Definition • Notation • Proof of formulas of Laplace Transformations	Lecture, Discussion	Quiz	CLO1, CLO2
2	Some important properties of Laplace Transformations • Linearity property • Change of scale property • Related mathematics	Discussion, Oral Presentation	Written Assignment	CLO1, CLO2
3	Multiplication by t power n Derivatives of Laplace Transformation • 1^{st} and 2^{nd} derivatives • General term derivatives • Related mathematics	Oral Presentation	Oral Presentation	CLO1, CLO2
4	Inverse Laplace transformation • Definition • Notation Proof of formulas of Laplace Transformations	Group Work	Group Assignment	CLO1, CLO2
5	Some important properties of Inverse	Case Study	Presentation	CLO1, CLO2

	Laplace Transformations • Linearity property • Change of scale property • Related mathematics			
6	Ordinary Differential Equations with Constant Coefficients • Initial and boundary Value problem • Related mathematics	Group Work	Quiz, Written Assignment	CLO1, CLO2
7	Applications to electrical circuits • L-R circuit • Related mathematics	Lecture, Discussion	Oral Presentation, Quiz	CLO5, CLO2
8	Fourier series • Definition • Periodic Function • Piecewise function • Odd and even functions • Related problems	Discussion, Oral Presentation	Group Assignment, Quiz	CLO3, CLO1
9	Some important properties of Fourier series	Oral Presentation	Presentation, Written Assignment	CLO3, CLO1

	• Dirichlet Conditions • Parseval's Identity • Theorems and related mathematics			
10	Fourier series of different types of function and Related mathematics and Orthogonality.	Oral Presentation	Quiz, Presentation	CLO3, CLO5
11	Half range Fourier Sine or Cosine Series • Definition • Related mathematics. Fourier integrals, Convolution Theorem, and mathematics.	Group Work	Written Assignment, Oral Presentation	CLO3, CLO5
12	Matrix • Definition • its operations and classifications. Symmetric and skew-Symmetric matrix • Definition • related theorems	Discussion, Oral Presentation	Group Assignment, Presentation	CLO4, CLO1
13	Symmetric and skew-Symmetric matrix • related mathematical problems Orthogonal, Involutary and Idempotent matrices • Definition	Discussion, Oral Presentation	Quiz, Group Assignment	CLO4, CLO1

	related mathematics			
14	Determinant and Inverse Matrix • Definition • Methodology • related mathematics	Oral Presentation	Written Assignment, Quiz	CLO4, CLO1
15	System of Linear Equations • Definition • Methodology • related mathematics Matrix method and Cramer's rule for solving linear systems. • Methodology • related mathematics	Lecture, Discussion	Oral Presentation, Group Assignment	CLO4, CLO1
16	Rank of a matrix and eigenvalue • Definition • related theorems • related mathematical problems	Practical Work	Presentation, Quiz	CLO4, CLO1
17	Cayley- Hamilton theorem • theorem • verification • mathematical problems Application of Linear Algebra in Engineering	Reading Assignment	Quiz, Written Assignment, Oral Presentation	CLO4, CLO1

REFERENCE BOOKS

1. Laplace Transformations– Schaum's Outline series.
2. Fourier Transformation- Schaum's Outline series.
3. Laplace and Fourier Transformations- Prof. Dr. Abdur Rahman.
4. Linear Algebra- Prof. Dr. Abdur Rahman
5. Linear Algebra- Haward Anton.
6. Linear Algebra- Schaum's Outline series

Assessment Pattern

CIE- Continuous Internal Evaluation (90 Marks)

Bloom's Category Marks (out of 60)	Tests (45)	Assignments (15)	Quizzes (15)	Attendance (15)
Remember	05			
Understand	05		05	
Apply	10	05	05	15
Analyze	10	05	05	
Evaluate	10	05		
Create	05			

SEE- Semester End Examination (60 Marks)

Bloom's Category	Test
Remember	10
Understand	10
Apply	10
Analyze	10
Evaluate	15
Create	5

Week 1
Topics: Laplace Transformation
Pages (8-10)

<u>Laplace Transformation</u>: Let *F(t)* be a function of t specified for *t>0*. Then the Laplace Transform of *F(t)*, denoted by $\mathcal{L}\{F(t)\}$ is define by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$,where the parameter s is real.

<u>Some formula of Laplace Transformation</u>

$$(i)\ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad (ii)\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a$$

$$(iii)\ \mathcal{L}\{cos\ at\} = \frac{s}{s^2 + a^2}, \quad s > 0 \qquad (iv)\ \mathcal{L}\{sin\ at\} = \frac{a}{s^2 + a^2}, \qquad s > 0$$

<u>Question:</u> Prove that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ s > a$

<u>Solution:</u>

Let, $F(t) = e^{at}$

By the definition of Laplace transformation, we know that

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

So, $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$

$$= \int_0^\infty e^{-st+at}\ dt \quad = \int_0^\infty e^{-(s-a)t}\ dt$$

$$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty \quad = \frac{1}{-(s-a)}(e^{-\infty} - e^0) \quad = \frac{1}{s-a},\ \ s > a$$

Question: Prove that $\mathcal{L}\{1\} = \frac{1}{s},\ s > 0$

Solution:

Let, $F(t) = 1$

By the definition of Laplace transformation, we know that

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\ dt$$

So, $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}.1\ dt$

$$= \int_0^\infty e^{-st}\ dt \quad = \left. \frac{e^{-st}}{-s} \right|_0^\infty \quad = \frac{1}{-s}(e^{-\infty} - e^0) \quad = \frac{1}{s},\ \ s > 0$$

Question: Prove that $\mathcal{L}\{t\} = \frac{1}{s^2},\ s > 0$

Solution:

Let, $F(t) = t$

By the definition of Laplace transformation, we know that

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\ dt$$

So, $\mathcal{L}\{t\} = \int_0^\infty e^{-st}.t\ dt$

$$= \int_0^\infty e^{-st} t\, dt \quad = t\int_0^\infty e^{-st}\, dt - \int_0^\infty \left\{ \frac{dt}{dt} \int_0^\infty e^{-st}\, dt \right\} dt$$

$$= t \left. \frac{e^{-st}}{-s} \right|_0^\infty - \int_0^\infty \left\{ 1.\frac{e^{-st}}{-s} \right\} dt = 0 - \left(\frac{1}{-s} \right) . \left. \frac{e^{-st}}{-s} \right|_0^\infty$$

$$= \frac{1}{-s^2}(e^{-\infty} - e^0) \quad = \frac{1}{s^2}, \ s > 0$$

Question: Prove that $\mathcal{L}\{4\,e^{at}\} = \frac{4}{s-a},\ s > a$

Question: Prove that $\mathcal{L}\{5\} = \frac{5}{s},\ s > 0$

Question: Prove that $\mathcal{L}\{3t\} = \frac{3}{s^2},\ s > 0$

Week 2

Topics: Properties of Laplace Transformation Pages (10-14)

Linear property of Laplace theorem: If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transformations $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F(t)\} = c_1f_1(s) + c_2f_2(s)$$

is called the linear property of Laplace Transformations. The result is easily extended for n terms as follows:

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t) + \cdots + c_nF_n(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\} + \cdots + c_n\mathcal{L}\{F_n(t)\}$$

Or,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} + \cdots + c_n\mathcal{L}\{f_n(t)\}$$

Questions-1: Find the Laplace transformation of $4e^{5t} + 6t^3 - 3\sin 4t + 5\cos 2t$

Solution: By linear property of Laplace transformations, we have

$$\mathcal{L}\{4e^{5t} + 6t^3 - 3\sin 4t + 5\cos 2t\}$$
$$= 4\mathcal{L}\{e^{5t}\} + 6\mathcal{L}\{t^3\} - 3\mathcal{L}\{\sin 4t\} + 5\mathcal{L}\{\cos 2t\}$$
$$= 4\left(\frac{1}{s-5}\right) + 6\left(\frac{3!}{s^4}\right) - 3\left(\frac{4}{s^2+4^2}\right) + 5\left(\frac{s}{s^2+2^2}\right)$$
$$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{5s}{s^2+4}$$

Question-2: Find the Laplace transformation of $3t^4 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$.

Solution: By linear property of Laplace transformations, we have

$$\mathcal{L}\{3t^4 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$$
$$= 3\mathcal{L}\{t^4\} + 4\mathcal{L}\{e^{-3t}\} - 2\mathcal{L}\{\sin 5t\} + 3\mathcal{L}\{\cos 2t\}$$
$$= 3\left(\frac{4!}{s^5}\right) + 4\left(\frac{1}{s+3}\right) - 2\left(\frac{5}{s^2+5^2}\right) + 3\left(\frac{s}{s^2+2^2}\right)$$
$$= \frac{72}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3\,s}{s^2+4}$$

Question: Find the Laplace transformation of the followings:

(i) $7t^4 + 5e^{-6t} - 4\sin 5t + 2\cos 2t$

(ii) $3t^3 + 4e^{-5t} + 3\cos 4t - 2\sin 6t$

(iii) $10t^6 - 15e^{10t} + 12\sin t + 6\cos 6t$

$(iv) 10\sin 10t - 12t^7 - 2\cos t + e^{-t}$

$(v) 10t^3 - 5e^{-7t} - 20\sin 6t + 20\cos 7t$

(vii) $13e^{10t} + 6e^{-t} + 12t^8 + 6\sin t + 2\cos 9t$

(viii) $2e^{-5t} + 7e^{6t} + t^7 - 2\sin 8t + 7\cos t$

Change of scale or shifting property:

If $\{F(t)\} = f(s)$, then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

Questions: Find the Laplace transformation of the expression$e^{-2t}(3\cos 6t - 5\sin 6t)$.

Solution: We have $\mathcal{L}\{3\cos 6t - 5\sin 6t\}$

$$= 3\left(\frac{s}{s^2+6^2}\right) - 5\left(\frac{6}{s^2+6^2}\right)$$

$$= \frac{3s}{s^2+36} - \frac{30}{s^2+36} = \frac{3s-30}{s^2+36}$$

Then $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = \frac{3(s+2)-30}{(s+2)^2+36} = \frac{3s-24}{s^2+4s+40}$

Question-2: Find the Laplace transformation of

$(i) e^{2t}(3\cos 4t - 4\sin 4t)$ $(ii) e^{-4t}(6\sin 3t - 5\cos 3t)$

Solution: (i) We have $\mathcal{L}\{3\cos 4t - 4\sin 4t\}$

$$= 3\left(\frac{s}{s^2+4^2}\right) - 5\left(\frac{4}{s^2+4^2}\right)$$

$$= \frac{3s}{s^2+16} - \frac{20}{s^2+16} = \frac{3s-20}{s^2+16}$$

Then $\mathcal{L}\{e^{2t}(3\sin 4t - 4\cos 4t)\} = \frac{3(s-2)-30}{(s-2)^2+36} = \frac{3s-6-30}{s^2-4s+4+36} = \frac{3s-6-3}{s^2-4s+40}$

(ii) We have $\mathcal{L}\{6\, sin3t - 5\cos 3t\}$

$$= 6\left(\frac{3}{s^2+3^2}\right) - 5\left(\frac{s}{s^2+3^2}\right)$$

$$= \frac{18}{s^2+9} - \frac{5s}{s^2+9} = \frac{18-5s}{s^2+9}$$

Then $\mathcal{L}\{e^{-4t}(6\, sin3t - 5\cos 3t)\} = \frac{18-5(s+4)}{(s+4)^2+9} = \frac{18-5s-20}{s^2+16s+16+9} = \frac{-(5s+2)}{s^2+16s+25}$

<u>Question:</u> Find the Laplace transformation of the followings:

(i) $2e^{-3t}(\sin 4t)$, $e^{-t}\cos 9t$

(ii) $e^{-3t}(6t^3 - 7\cos t + 4\sin t)$

(iii) $e^{-4t}(6\sin 3t - 5\cos 3t)$

(iv) $e^{-t}(\cos 2t + 2t^2)$

(v) $e^{4t}(3t^4 + 2\sin 7t + \cos 5t)$

(vi) $e^{-5t}(4t^3 + 3\cos t - 4\sin 6t)$

(vii) $e^{3t}(7\cos 3t + 4\sin 6t)$

Week 3
Topics: Laplace Transformation of derivatives
Pages (13-15)

Laplace Transformation of derivatives:

If $\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$

and $\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

Multiplication by powers of t:

If $\{F(t)\} = f(s)$, then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

Question: Find $\mathcal{L}\{t\ sinat\}$.

Solution: Since we know that,

$$\mathcal{L}\{sinat\} = \frac{a}{s^2 + a^2}$$

So,

$$\mathcal{L}\{t\ sinat\} = \{(-1)^1\} \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2 + a^2)^2}$$

Question: Find $\mathcal{L}\{t^2 e^{2t}\}$ *and* $\mathcal{L}\{t\ cosat\}$.

Week 4
Topics: Inverse Laplace Transformation Pages (15-17)

Inverse Laplace Transformation: If the Laplace transformation of a function *F(t)* is *f(s)*, i.e $\mathcal{L}\{F(t)\} = f(s)$, then *F(t)* is called the inverse Laplace transformation of *f(s)*, and we can write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$, where $\mathcal{L}^{-1}$ is called the inverse Laplace operator.

Some formula of inverse Laplace Transformation

$$(i)\mathcal{L}^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!} \qquad (ii)\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$$

$$(iii)\mathcal{L}^{-1}\{\frac{s}{s^2+a^2}\} = cos\ at \qquad (iv)\mathcal{L}^{-1}\{\frac{a}{s^2+a^2}\} = sin\ at$$

Linear property of inverse Laplace transformation: If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are functions with Laplace transformations $F_1(t)$ and $F_2(t)$ respectively, then

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\}$$

is called the linear property of inverse Laplace transformation. And for n times we can write,

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s) + \cdots + c_n f_n(s)\}$$
$$= c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\} + \cdots + c_n\mathcal{L}^{-1}\{f_n(s)\}$$

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s) + \cdots + c_n f_n(s)\}$$
$$= c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\} + \cdots + c_n\mathcal{L}^{-1}\{f_n(s)\}$$

Questions-1: Find the inverse Laplace transformation of the expression $\frac{1}{s^5}+\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}$

Solution: By linear property of Laplace transformations, we have

$$\mathcal{L}^{-1}\{\frac{1}{s^5}+\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}\}$$

$$=\mathcal{L}^{-1}\{\frac{1}{s^5}\}+4\mathcal{L}^{-1}\{\frac{1}{s-2}\}-\mathcal{L}^{-1}\{\frac{3s}{s^2+16}\}+\mathcal{L}^{-1}\{\frac{5}{s^2+4}\}$$

$$=\frac{1}{24}\mathcal{L}^{-1}\{\frac{4!}{s^5}\}+4\mathcal{L}^{-1}\{\frac{1}{s-2}\}-3\mathcal{L}^{-1}\{\frac{s}{s^2+16}\}+\frac{5}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\}$$

$$=\frac{1}{24}t^4+4e^{2t}-3\cos 4t+\frac{5}{2}\sin 2t$$

Question-2: Find the inverse Laplace transformation $\frac{5s+4}{s^5}+\frac{6}{s-3}-\frac{8s}{s^2+9}+\frac{7}{s^2+25}$

Solution: By linear property of Laplace transformations, we have

$$\mathcal{L}^{-1}\{\frac{5s+4}{s^5}+\frac{6}{s-3}-\frac{8s}{s^2+9}+\frac{7}{s^2+25}\}$$

$$=5\,\mathcal{L}^{-1}\{\frac{1}{s^4}\}+4\,\mathcal{L}^{-1}\{\frac{1}{s^5}\}+6\mathcal{L}^{-1}\{\frac{1}{s-3}\}-\mathcal{L}^{-1}\{\frac{8s}{s^2+9}\}+\mathcal{L}^{-1}\{\frac{7}{s^2+25}\}$$

$$=\frac{5}{6}\mathcal{L}^{-1}\{\frac{3!}{s^{3+1}}\}+\frac{4}{24}\mathcal{L}^{-1}\{\frac{4!}{s^{4+1}}\}+6\{\frac{1}{s-3}\}-8\mathcal{L}^{-1}\{\frac{s}{s^2+9}\}+\frac{7}{5}\mathcal{L}^{-1}\{\frac{5}{s^2+25}\}$$

$$=\frac{5}{6}t^3+\frac{1}{6}t^4+6\,e^{3t}-8\cos 3t+\frac{7}{5}\sin 5t$$

Question: Find the Inverse Laplace transformation of the followings:

$(i)\ \frac{6}{2s-3}-\frac{3+4s}{9s^2+16}+\frac{8-6s}{16s^2+9}$

$(ii)\ \frac{1}{s^5}+\frac{4}{s-2}+\frac{3s}{s^2+16}+\frac{5}{s^2+4}$

$(iii)\ \frac{5s+4}{s^5}+\frac{6}{s-3}-\frac{8s}{s^2+9}+\frac{7}{s^2+25}$

$(iv)\ \frac{3}{s+4}-\frac{2s+5}{s^2+16}$

$(v)\ \frac{s+10}{s^4}-\frac{4}{s-6}+\frac{s+8}{s^2+4}$

Change of scale property:

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$

Questions: Find the inverse Laplace transformation of the expression $\frac{6s-4}{s^2-4s+20}$.

Solution: We have $\mathcal{L}^{-1}\{\frac{6s-4}{s^2-4s+20}\} = \mathcal{L}^{-1}\{\frac{6s-12+8}{s^2-2.s.2+2^2+16}\} = \mathcal{L}^{-1}\{\frac{6(s-2)+8}{(s-2)^2+16}\}$

$$= 6\,\mathcal{L}^{-1}\{\frac{(s-2)}{(s-2)^2+16}\} + 2\,\mathcal{L}^{-1}\{\frac{4}{(s-2)^2+16}\}$$

$$= 6\,e^{2t}\cos 4t + 2\,e^{2t}\sin 4t$$

Question-2: Find the inverse Laplace transformation of $(i)\ \frac{4s-25}{s^2-6s+34}$ $(ii)\ \frac{6s-4}{s^2-2s+10}$

Solution: We have $\mathcal{L}^{-1}\{\frac{4s-25}{s^2-6s+34}\} = \mathcal{L}^{-1}\{\frac{4s-12-13}{s^2-2.s.3+3^2+25}\} = \mathcal{L}^{-1}\{\frac{4(s-3)-13}{(s-3)^2+5^2}\}$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{(s-3)}{(s-3)^2+5^2}\right\} - \frac{13}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s-3)^2+5^2}\right\}$$

$$= 4\ e^{3t}\cos 5t - \frac{13}{5}\ e^{3t}\sin 5t$$

Question: Find the Inverse Laplace transformation of the followings:

$$(i)\frac{6s-10}{s^2-4s+20} \qquad (ii)\frac{4s+12}{s^2+8s+16}$$

$$(iii)\frac{3s-8}{4s^2-25} \qquad (iv)\frac{2s+4}{s^2+2s+5}$$

Week 5

Topics: Initial and boundary

Value problem Pages (18-20)

Question: Solve $Y'+Y=t,\ \ Y(0)=1,\ \ Y'(0)=-2.$

Solution: Given that, $Y''+Y=t$

Taking the Laplace transformation on both sides of the differential equation and using the given conditions, we have $\mathcal{L}\{Y''\}+\mathcal{L}\{Y'\}=\mathcal{L}\{t\}$

$$\Rightarrow s^2 y - sY(0) - Y'(0) + y = \frac{1}{s^2}$$

$$\Rightarrow s^2 y - s + 2 + y = \frac{1}{s^2}$$

$$\Rightarrow y = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

Again taking Inverse Laplace Transformations, we have

$$\mathcal{L}^{-1}\{y\} = Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\}$$

$$Y = t + cost - 3sint$$

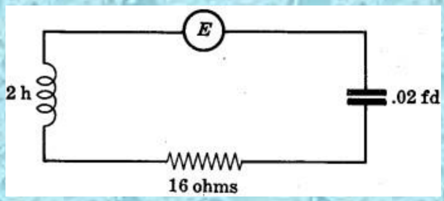
Week 6

Topics: L-R Circuit problem

Pages (19-22)

Question: An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of 0.02 farads are connected in series with an electromotive force (e.m.f) of 300 volts. At $t = 0$ the charge on the capacitor and current in the circuit is zero. Find the charge and current at any time $t > 0$.

Solution: Let Q and I be the instantaneous charge and current respectively at time t.



By Kirchhoff's Laws, we have $2\frac{dI}{dt}+16I+\frac{Q}{0.02}=300 \quad \cdots\cdots \quad (1)$

Since $I=\frac{dQ}{dt}$, so (1) becomes $\quad 2\frac{d^2Q}{dt^2}+16\frac{dQ}{dt}+50Q=300$

$$or,\quad \frac{d^2Q}{dt^2}+8\frac{dQ}{dt}+25Q=150 \quad \cdots\cdots \quad (2)$$

with the initial conditions $Q(0)=0, \quad I(0)=Q'(0)=0$

Taking Laplace transformation in (2), we find

$$\mathcal{L}\{\frac{d^2Q}{dt^2}+8\frac{dQ}{dt}+25Q\}=\mathcal{L}\{150\}$$

$$\Rightarrow \left\{s^2q-sQ(0)-Q'(0)\right\}+8\left\{sq-Q(0)\right\}+25q=\frac{150}{s}$$

$$\Rightarrow s^2q+8sq+25q=\frac{150}{s}$$

$$\Rightarrow \left(s^2+8s+25\right)q=\frac{150}{s}$$

$$\Rightarrow q=\frac{150}{s\left(s^2+8s+25\right)}$$

$$\Rightarrow q=\frac{6}{s}-\frac{6s+48}{s^2+8s+25}=\frac{6}{s}-\frac{6(s+4)+24}{(s+4)^2+9}$$

$$\therefore q=\frac{6}{s}-\frac{6(s+4)}{(s+4)^2+3^2}-\frac{24}{(s+4)^2+3^2} \quad \cdots\cdots \quad (3)$$

Taking inverse Laplace transformation in (3), we get

$$\mathcal{L}^{-1}\{q\}=\mathcal{L}^{-1}\left\{\frac{6}{s}-\frac{6(s+4)}{(s+4)^2+3^2}-\frac{24}{(s+4)^2+3^2}\right\}$$

$$\therefore Q = 6 - 6\, e^{-4t} \cos 3t - 8\, e^{-4t} \sin 3t$$

The current of the circuit is

$$I = \frac{dQ}{dt} = 24e^{-4t} \cos 3t + 32\, e^{-4t} \sin 3t + 18\, e^{-4t} \sin 3t - 24\, e^{-4t} \cos 3t = 50\, e^{-4t} \sin 3t$$

Question: An inductor of 3 henrys, a resistor of 30 ohms and an electromotive force (e.m.f) of 150

volts. At $t = 0$ the current in the circuit is zero. Find the current at any time, $t>0$.

Question: An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of 0.02 farads are connected in series with an electromotive force (e.m.f) of 100 volts. At $t = 0$ the charge on the capacitor and current in the circuit is zero. Find the charge and current at any time $t > 0$.

Week 7

Topics: Fourier analysis

Pages (22-24)

Some Important Functions:

Periodic Functions:

A function *f(x)* is said to have a period *P* or to be periodic with periodic *P* if for all *x*, $f(x+P)=f(x)$, where *P* is a positive constant. The least value of $P>0$ is called the least period or simply the period of *f(x)*.

Ex1: The functions *sinx* and *cosx* has periods 2π, 4π, 6π, ….. However, 2π is the least period or periods of *sinx* and *cosx.*

Ex2: The period of *tanx* is π.

Ex.3: A costant has any positive number as a period.

Piecewise Continuous Functions:

A function *f(x)* is said to be piecewise continuous in an interval (i) the interval can be divided into a finite number of subintervals in each of which *f(x)* is continuous and (ii) the limits of *f(x)* as *x* approaches the endpoints of each subintervals are finite.

Ex: $f(x)=|x|=\begin{cases} x, & \text{if } x\geq 0 \\ -x, & \text{if } x\leq 0 \end{cases}$ is a piecewise continuous function.

Even and Odd Functions:

A function *f(x)* is called even function if $f(-x)=f(x)$ and is called odd function if $f(-x)=-f(x)$.

Ex: x^2, x^4, x^6, *cosx, secx* are even functions.

Ex: x^3, x^5, x^7, *sinx, tan3x* are odd functions.

Fourier Series:

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L)=f(x)$, i.e. assume that $f(x)$ has the period $2L$. The fourier series or fourier expansion corresponding to $f(x)$ is defined to be

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

Where the fourier coefficients a_0, a_n and b_n are

$$\begin{cases} a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\dfrac{n\pi x}{L}\,dx \qquad ,n = 0,1,2........ \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\dfrac{n\pi x}{L}\,dx \end{cases}$$

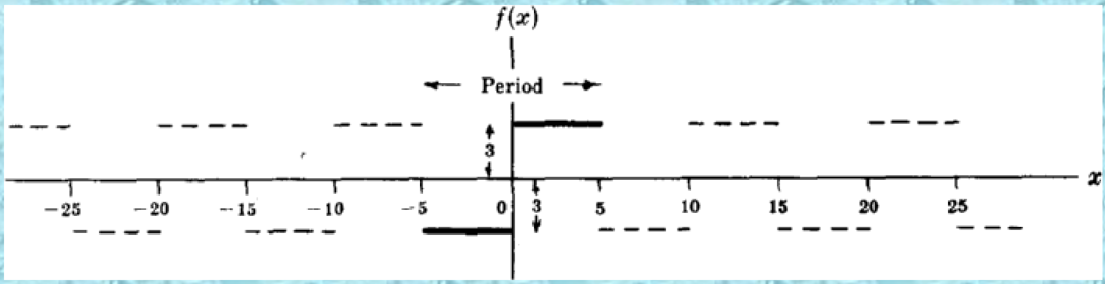
Problems: Graph each of the following functions:

$$(a)\ f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases} \qquad Period = 10$$

$$(b)\ f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad Period = 2\pi$$

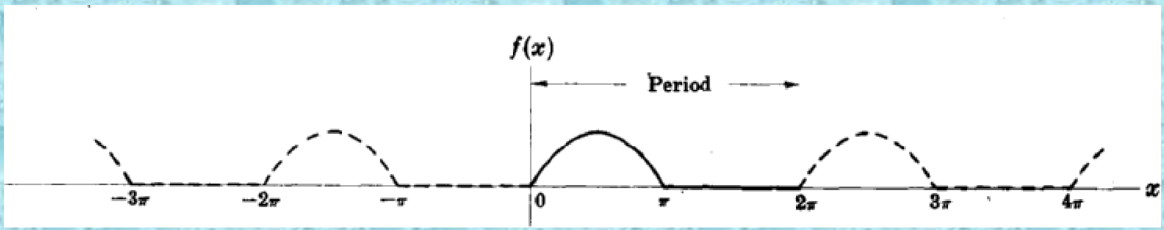
$$(c)\ f(x) = \begin{cases} 0 & 0 \le x < 2 \\ 1 & 2 \le x < 4 \\ 0 & 4 \le x < 6 \end{cases} \qquad Period = 6$$

Solution: (a)



Since the period is 10, the portion of the graph in $-5<x<5$ which is indicated heavily is extended periodically outside of the range which is indicated in dashed. It is noted that $f(x)$ is not defined at $x=0, 5, -5, 10, -10, 15, -15, 20$…These are the discontinuous point of $f(x)$.

(b)



Since the period is 2π, the portion of the graph in $0<x<2\pi$ which is indicated heavily is extended periodically outside of the range which is indicated in dashed. It is noted that $f(x)$ is defined for all x, and is continuous everywhere.

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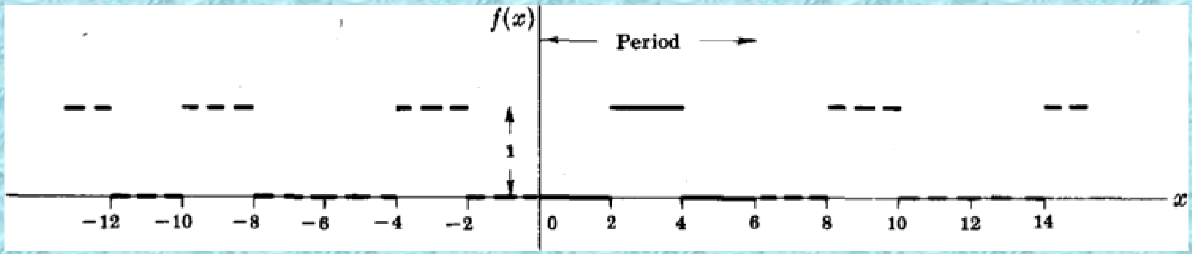


Since the period is 6, the portion of the graph in $0<x<6$ which is indicated heavily is extended periodically outside of the range which is indicated in dashed. It is noted that $f(x)$ is defined for all x, and $x=-2, 2, -4, 4, -8, 8$ …are the discontinuous point of $f(x)$.s

Week 8

Topics: Graph of functions

Pages (25-27)

Problems:

Classify each of the following functions according as they are even, odd or neither even nor odd.

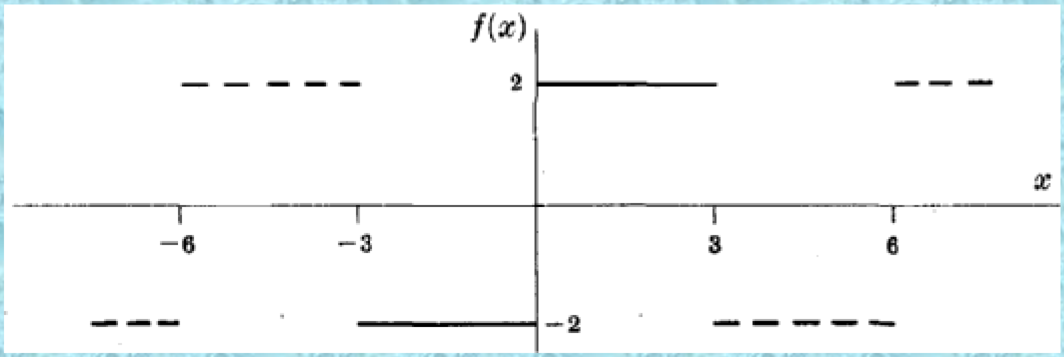
$$(a)\ f(x)=\begin{cases} 2 & 0<x<3 \\ -2 & -3<x<0 \end{cases} \quad Period = 6$$

$$(b)\ f(x)=\begin{cases} \cos x & 0\le x\le \pi \\ 0 & \pi < x < 2\pi \end{cases} \quad Period = 2\pi$$

$$(c)\ f(x)=x(10-x), \quad 0<x<10 \quad Period = 6$$

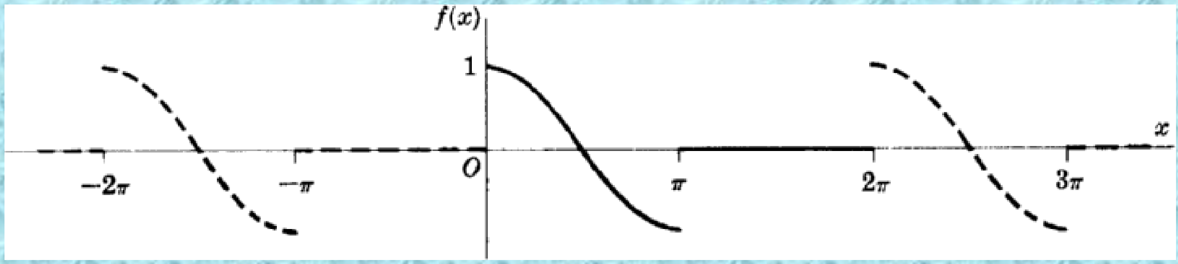
Solution:

(a) The graphical representation of the given function is



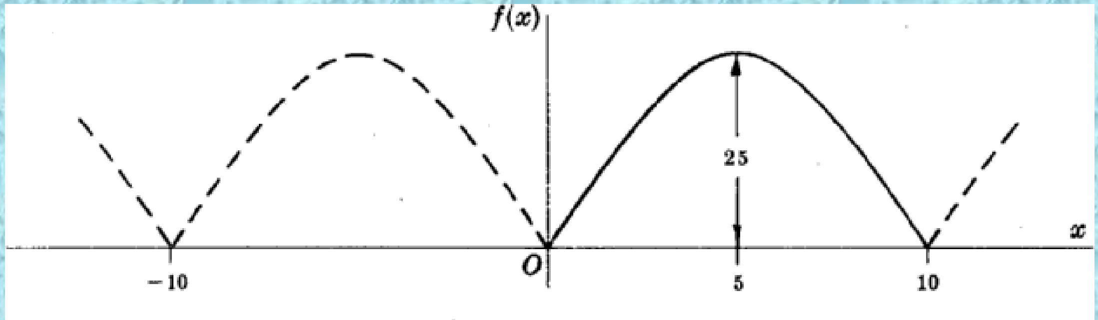
From the above figure we can see that the function is symmetric about the origin. So, it is seen from the figure that *f(-x)=-f(x)*, Hence the function is odd.

(b) The graphical representation of the given function is as follows



From the above figure we can see that the function is neither even nor odd.

© The graphical representation of the given function is



From the above figure we can see that the function is symmetric about y-axis. So, it is seen from the figure that *f(-x)=f(x)*, Hence the function is even.

<u>Exercise1:</u> Graph each of the following functions and classify them according as they are even, odd or neither even nor odd.

$(a)\ f(x)=\begin{cases}8 & 0<x<2\\ -8 & 2<x<4\end{cases} \quad Period=4$ $\qquad (b)\ f(x)=\begin{cases}-x & -4\le x\le 0\\ x & 0\le x\le 4\end{cases} \quad Period=8$

$(c)\ f(x)=4x,\quad 0<x<10 \quad Period=10$ $\qquad (d)\ f(x)=\begin{cases}2x & 0\le x\le 3\\ 0 & -3\le x<0\end{cases} \quad Period=6$

Week 9

Topics: Dirichlet condition, Parsival's Identity

Pages (28-32)

Dirichlet conditions for Fourier series

A set of Dirichlet conditions for the convergence of Fourier series are:

(1) a function "f" must be absolutely integrable over a period.

(2) a function "f" has bounded variation over one time period. The functions with bounded variations can have

(i) at most a countably infinite number of maxima and minima, and

(ii) at most a countably infinite number of finite discontinuities.

Dirichlet conditions for Fourier transform

A set of Dirichlet conditions for the convergence of Fourier transform are:

(1) a function "f" is absolutely integrable over the entire duration of time.

(2) a function "f" has bounded variation over the entire duration of time. The functions with bounded variations can contain (i) at most a countably infinite

number of maxima and minima, and (ii) at most a countably infinite number of finite discontinuities.

Dirichlet conditions are sufficient but not necessary conditions.

Parseval's Identity: Let the Fourier series corresponding to *f(x)* converges

uniformly in *(-L,L)*, then the Parsival's Identity is

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n{}^2+b_n{}^2\right)$$

Where a_0, a_n and b_n are Fourier coefficients respectively.

Parseval's Identity: Let the Fourier series corresponding to *f(x)* converges

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Where a_0, a_n and b_n are Fourier coefficients respectively.

Proof:

$$If\quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\text{............(1)}$$

Then multiplying (1) by *f(x)* and integrating term by term from –L to L we get

$$\int_{-L}^{L}\{f(x)\}^2 = \frac{a_0}{2}\int_{-L}^{L} f(x) + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\right)$$

$$= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$So,\ \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Where we have used the results

$$\int_{-L}^{L} f(x)dx = La_0,\ \int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = La_n,\ \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = Lb_n \text{......(2)}$$

Is obtained from the Fourier Coefficients.

Hence the Parsival's identity is proved.

Problem: (a) Expand *f(x)=x, 0<x<2* in a half range cosine series.(b) Write

Parsival's Identity corresponding to the Fourier series of (a).(c) Determine fom (b)

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \text{.....} + \frac{1}{n^4} + \ldots$$

the sum S of the series

Solution:

(a) Extend the definition of the given function to that of the even function of period 4 which is shown in the below figure. This is sometimes called the even extension of *f(x)*. Then 2*L*=4, *L*=2.

Thus $b_n=0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\frac{\sin n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n^2\pi^2}(\cos n\pi - 1) \text{ if } n \neq 0$$

$$If \quad n = 0, \quad a_0 = \int_0^2 x dx = 2$$

$$Then\ f(x) = 1 + \sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + ...\right)$$

(b) From (a) we get,

$$L = 2, a_0 = 2, a_n = \frac{4}{n^2\pi^2}(\cos n\pi - 1), n \neq 0; b_n = 0$$

Then Parseval's identity becomes

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2 = \frac{1}{2}\int_{-2}^{2}x^2 dx = \frac{2^2}{2} + \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi - 1)$$

$$or \quad \frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ...\right)$$

$$i.e \quad \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\pi^4}{96}$$

© Here,

$$s = \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\right)$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\ldots\right)$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\right)$$

$$= \frac{\pi^4}{96} + \frac{S}{16}$$

$i.e$ from which, $S = \frac{\pi_4}{90}$

<u>Exercise5:</u> (a) Expand *f(x)=x, 0<x<2* in a half range sine series.(b) Write

Parsival's Identity corresponding to the Fourier series of (a).

<u>Exercise6:</u> (a) Expand *f(x)=x, 0<x<4* in a half range cosine series.(b) Write

Parsival's Identity corresponding to the Fourier series of (a).

<u>Exercise7:</u> (a) Expand *f(x)=x, 0<x<4* in a half range sine series.(b) Write

Parsival's Identity corresponding to the Fourier series of (a).

Orthogonality:

Orthogonality is a fundamental concept in Fourier series, which are used to break down periodic functions into simpler terms:

- **Definition**: Two functions are orthogonal on an interval if their inner product is zero. The inner product is defined as the integral of the product of the two functions over the interval:

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0$$

Orthogonal sets

A set of functions is orthogonal if any two functions in the set are orthogonal. For example, the set of functions $(1, cosx, cos2x, cos3x)$ is orthogonal in the interval (-π π).

Week 10

Topics: Fourier Series Related mathematics

Pages (32-35)

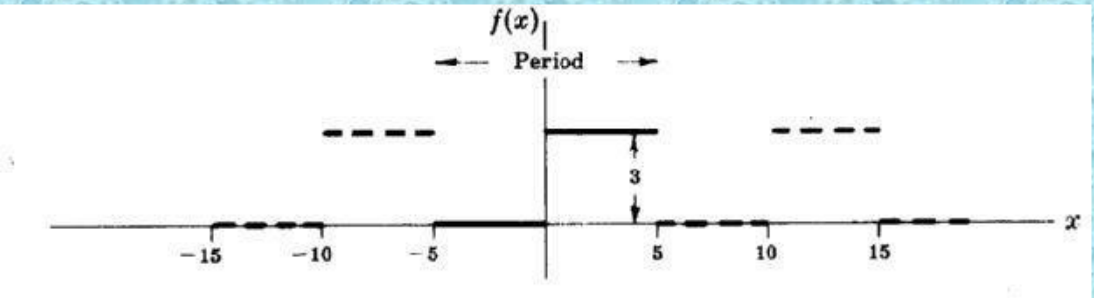
Problem:

(a) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \quad Period = 10$$

(b) Write the corresponding Fourier series.

Solution: The graph of *f(x)* is as follows



(a) Period$=2L=10$ and $L=5$. Choose the interval c to $c+2L$ as -5 to 5, so that $c=-5$. Then

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0} 0.\cos\frac{n\pi x}{5}dx + \int_0^5 3.\cos\frac{n\pi x}{5}dx\right\}$$

$$= \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}dx = \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \quad if \;\; n \neq 0$$

$$If \;\; n=0,\; a_n = a_0 = \frac{1}{5}\int_0^5 3.\cos\frac{n\pi .0}{5}dx = \frac{3}{5}\int_0^5 3.dx = 3$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0} 0.\sin\frac{n\pi x}{5}dx + \int_0^5 3.\sin\frac{n\pi x}{5}dx\right\}$$

$$= \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}dx = \frac{3}{5}\left(-\frac{5}{n\pi}s\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

(b) The corresponding Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{L}+b_n Sin\frac{n\pi x}{L}\right)$$

$$=\frac{3}{2}+\sum_{n=1}^{\infty}\left(\frac{3}{n\pi}(1-Cos n\pi)Sin\frac{n\pi x}{5}\right)$$

$$=\frac{3}{2}+\frac{6}{\pi}\left(\sin\frac{\pi x}{5}+\frac{1}{3}\sin\frac{3\pi x}{5}+\frac{1}{5}\sin\frac{5\pi x}{5}+........\right)$$

Exercise2: Graph each of the following functions, find the Fourier coefficients corresponding to the functions and its corresponding Fourier series.

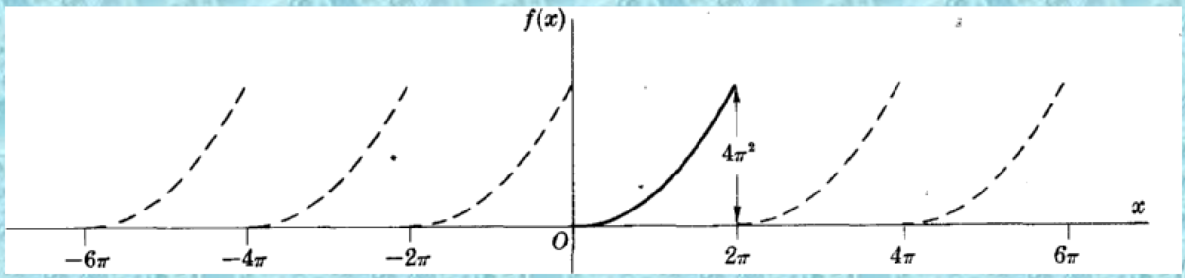
(i) $f(x)=\begin{cases}0 & -5<x<0\\ 3 & 0<x<5\end{cases}$ $Period=10$ (ii) $f(x)=\begin{cases}8 & -2<x<0\\ 0 & 0<x<2\end{cases}$ $Period=4$

(iii) $f(x)=\begin{cases}0 & -3<x<0\\ 4 & 0<x<3\end{cases}$ $Period=6$ (iv) $f(x)=\begin{cases}-2 & -3<x<0\\ 2 & 0<x<3\end{cases}$ $Period=6$

(v) $f(x)=\begin{cases}-3 & -5<x<0\\ 3 & 0<x<5\end{cases}$ $Period=10$ (vi) $f(x)=\begin{cases}8 & -4<x<0\\ -8 & 0<x<4\end{cases}$ $Period=8$

Problem2: Expand $f(x)=x^2$, $0<x<2\pi$, in a Fourier series if the period is 2π.

Solution: The graph of $f(x)$ with period 2π is as follows



Period=$2L=2\pi$ and $L=\pi$. Choosing c=0, we have

$$a_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\frac{n\pi x}{L}\ dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos n\pi\ dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{sinnx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2}, n \neq 0$$

$$If\ \ n = 0,\ a_n = a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\frac{n\pi x}{L}\ dx = \frac{1}{\pi}\int_{0}^{2\pi} x^2 sinn\pi\ dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

$$Then\ f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)\ for\ \ 0 < x < 2\pi$$

Exercise3: Graph each of the following functions, and also find its corresponding Fourier series.

(i) $f(x) = 2x^2,\ 0 < x < 2\pi$

(ii) $f(x) = ax^2,\ 0 < x < 2\pi$, *where a is any arbitrary* constant.

(iii) $f(x) = x^2,\ \ 0 < x < \pi$

Week 11

Topics: Half range Fourier Series and applications

Pages (35-38)

Half range Fourier sine or cosine Series:

A half-range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half-range series corresponding to a given function is desired, the function is generally defined in the interval *(0, L)* and then the function is specified as odd or even. In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx \quad \text{for half-range sine series} \\ b_n = 0, \quad a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\dfrac{n\pi x}{L}\,dx \quad \text{for half-range cosine series} \end{cases}$$

Problem3: Expand $f(x)=x,\ 0<x<2$, in a half range (a) sine series, (b) cosine series.

Solution:

(a) Extend the definition of the given function to that of the odd function of period 4 which is shown in the below figure. This is sometimes called the odd extension of $f(x)$. Then $2L=4$, $L=2$.

Thus $a_n=0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(-\frac{2}{n\pi}\frac{\cos n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

$$Then\ f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\,\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \ldots\ldots\right)$$

(b) Extend the definition of the given function to that of the even function of period 4 which is shown in the below figure. This is sometimes called the even extension of *f(x)*. Then 2*L*=4, *L*=2.

Thus $b_n=0$,

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$=\left\{(x)\left(\frac{2}{n\pi}\frac{\sin n\pi x}{2}\right)-(1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2=\frac{-4}{n^2\pi^2}(\cos n\pi-1)\text{ if } n\neq 0$$

$$If\quad n=0,\ \ a_0=\int_0^2 xdx=2$$

$$Then\ f(x)=1+\sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}(\cos n\pi-1)\cos\frac{n\pi x}{2}$$

$$=1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+.......\right)$$

It should be noted that although both series of (a) and (b) represent *f(x)* in the interval $0<x<2$, the second series converge more rapidly.

Exercise 4: Expand the followings functions in a half range (a) sine series, (b) cosine series.

$(i)\ f(x)=4x,\quad 0<x<4$

$(i)\ f(x)=ax,\quad 0<x<2\quad$ *where a is any arbitrary constant.*

$(i)\ f(x)=x,\quad 0<x<\pi$

Application:

Problem: Solve the boundary value problem

$$\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2},\ u(0,t)=10,\ \ u(3,t)=40,\ u(x,0)=25,\ \ |u(x,t)|<M$$

Solution: To solve the present problem assume that $u(x,t)=v(x,t)+\phi(x,t)$ where $\phi(x,t)$ is to be suitably determined. In terms of $v(x,t)$ the boundary value problem becomes

$$\frac{\partial v}{\partial t}=2\frac{\partial^2 v}{\partial x^2}+2\phi''(x),\ v(0,t)+\phi(0)=10,\ \ v(3,t)+\phi(3)=40,\ v(x,0)+\phi(x)=25,\ \ |v(x,t)|<M$$

This can be simplified by choosing

$$\phi''(x)=0,\ \ \phi(0)=3,\ \ \phi(3)=40$$

From which e can find $\phi(x)=10x+10$, o that the resulting boundary value problem is

$$\frac{\partial v}{\partial t}=2\frac{\partial^2 v}{\partial x^2},\ \ v(0,t)=10,\ \ v(3,t)=40,\ v(x,0)=15-10x$$

We can find the solution of this problem is in the form

$$v(x,t) = \sum_{m=1}^{\infty} B_m e^{-2m^2\pi^2 t/9} \sin\frac{m\pi x}{3}$$

The last condition yields

$$15 - 10x = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{3}$$

From which

$$B_m = \frac{2}{3}\int_0^3 (15 - 10x)\sin\frac{m\pi x}{3}dx = \frac{30}{m\pi}(\cos m\pi - 1)$$

Since $u(x,t) = v(x,t) + \phi(x,t)$, we have finally

$$u(x,t)=10x+10+\sum_{m=1}^{\infty}\frac{30}{m\pi}(\cos m\pi - 1)e^{-2m^2\pi^2 t/9}\sin\frac{m\pi x}{3}$$

as the required solution.

The term *10x+10* is the *steady-state* temperature, i.e. the temperature after a long time has elapsed.

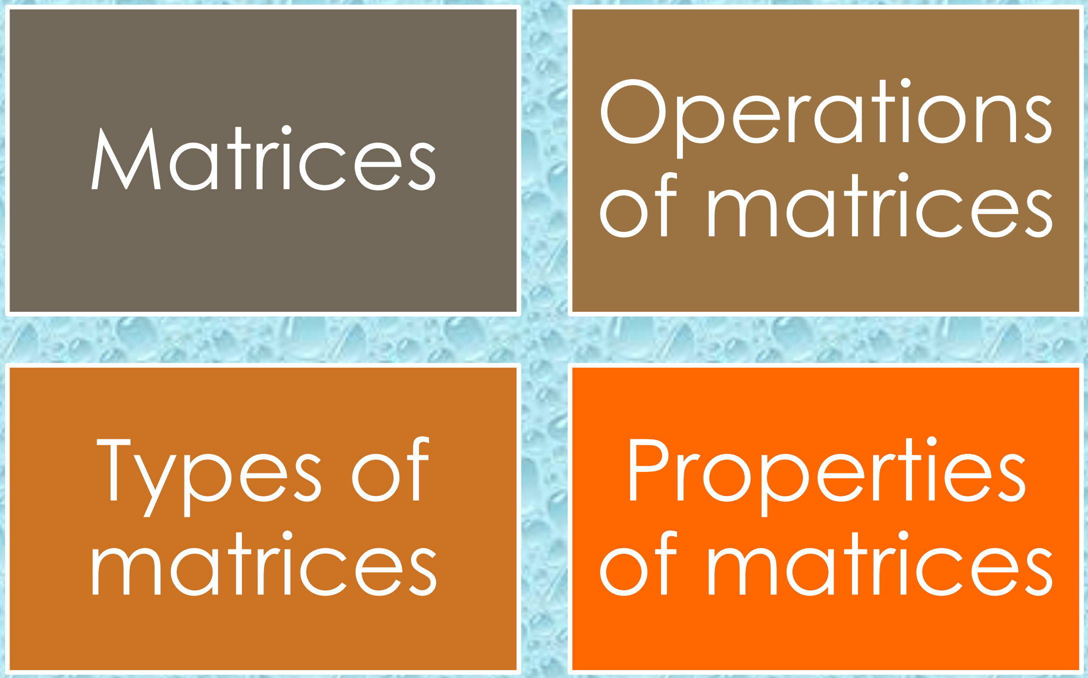
Week 12

Topic: Matrix and its operations

Page no. (39-70)

Matrix

A matrix is an array of numbers arranged in the form of rows and columns. The number of rows and columns of a matrix are known as its dimensions which is given by $m \times n$, where m and n represent the number of rows and columns respectively. Apart from basic mathematical operations, there are certain elementary operations



Matrices

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

Both *A* and *B* are examples of matrix. A matrix is a rectangular array of numbers enclosed by a pair of bracket.

Why matrix?

Matrices

In the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

- numbers a_{ij} are called *elements*. First subscript indicates the row; second subscript indicates the column. The matrix consists of mn elements
- It is called "the $m \times n$ matrix $A = [a_{ij}]$" or simply "the matrix A " if number of rows and columns are understood.

Matrices

Square matrices

- When $m = n$, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

- A is called a "square matrix of order n" or "n-square matrix"
- elements a_{11}, a_{22}, a_{33},…., a_{nn} called diagonal elements.
- $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$ is called the *trace* of A.

Matrices

Equal matrices

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal ($A = B$) iff each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$.
- *iff* pronouns "if and only if"

  if $A = B$, it implies $a_{ij} = b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$;

  if $a_{ij} = b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$, it implies $A = B$.

Equal matrices

Example: $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given that $A = B$, find a, b, c and d.

if $A = B$, then $a = 1$, $b = 0$, $c =$ -4 and $d = 2$.

Matrices

Zero matrices

- Every element of a matrix is zero, it is called a zero matrix, i.e.,

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

Operations of matrices

Sums of matrices

- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then $A + B$ is defined as a matrix $C = A + B$, where $C = [c_{ij}]$, $c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$.

Example: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

Evaluate $A + B$ and $A - B$.

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Operations of matrices

Sums of matrices

- Two matrices of the same order are said to be *conformable* for addition or subtraction.
- Two matrices of different orders cannot be added or subtracted, e.g.,

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

are NOT conformable for addition or subtraction.

Operations of matrices

Scalar multiplication

- Let λ be any scalar and $A = [a_{ij}]$ is an $m \times n$ matrix. Then $\lambda A = [\lambda a_{ij}]$ for $1 \le i \le m, 1 \le j \le n$, i.e., each element in A is multiplied by λ.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$. Evaluate $3A$.

$$3A = \begin{bmatrix} 3\times 1 & 3\times 2 & 3\times 3 \\ 3\times 0 & 3\times 1 & 3\times 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 12 \end{bmatrix}$$

- In particular, $\lambda = -1$, i.e., $-A = [-a_{ij}]$. It's called the *negative* of A. Note: $A - A = 0$ is a zero matrix

Operations of matrices

Properties

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Matrices A, B and C are conformable,

- $A + B = B + A$ (commutative law)
- $A + (B + C) = (A + B) + C$ (associative law)
- $\lambda(A + B) = \lambda A + \lambda B$, where λ is a scalar (distributive law

Operations of matrices

Properties

Example: Prove $\lambda(A + B) = \lambda A + \lambda B$. Let $C = A + B$, so $c_{ij} = a_{ij} + b_{ij}$.

Consider $\lambda c_{ij} = \lambda\ (a_{ij} + b_{ij}) = \lambda a_{ij} + \lambda b_{ij}$, we have,

$\lambda C = \lambda A + \lambda B$.

Since $\lambda C = \lambda(A + B)$, so $\lambda(A + B) = \lambda A + \lambda B$

Operations of matrices

Matrix multiplication

- If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then AB is defined as a $m \times n$ matrix $C = AB$, where $C = [c_{ij}]$ with $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$ for $1 \le i \le m$, $1 \le j \le n$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = AB$. Evaluate c_{21}.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \qquad c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22$$

Operations of matrices

Matrix multiplication

Example: $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1\times(-1) + 2\times 2 + 3\times 5 = 18 \\ c_{12} = 1\times 2 + 2\times 3 + 3\times 0 = 8 \\ c_{21} = 0\times(-1) + 1\times 2 + 4\times 5 = 22 \\ c_{22} = 0\times 2 + 1\times 3 + 4\times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix} = \begin{bmatrix}18 & 8\\ 22 & 3\end{bmatrix}$$

Operations of matrices

Matrix multiplication

- In particular, A is a $1 \times m$ matrix and

B is a $m \times 1$ matrix, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

then $C = AB$ is a scalar. $C = \sum_{k=1}^{m} a_{1k} b_{k1} = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1m}b_{m1}$

Operations of matrices

Matrix multiplication

- BUT BA is a $m \times m$ matrix!

$$BA = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & b_{11}a_{12} & \dots & b_{11}a_{1m} \\ b_{21}a_{11} & b_{21}a_{12} & & b_{21}a_{1m} \\ \vdots & & \ddots & \\ b_{m1}a_{11} & b_{m1}a_{12} & & b_{m1}a_{1m} \end{bmatrix}$$

- So $AB \neq BA$ in general !

Operations of matrices

Properties

Matrices A, B and C are conformable,

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $A(BC) = (AB) C$
- $AB \neq BA$ in general
- $AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$
- $AB = AC$ NOT necessarily imply $B = C$

However

Operations of matrices

Properties

Example: Prove $A(B + C) = AB + AC$ where A, B and C are n-square matrices

Let $X = B + C$, so $x_{ij} = b_{ij} + c_{ij}$. Let $Y = AX$, then

$$y_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n}(a_{ik} b_{kj} + a_{ik} c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

So $Y = AB + AC$; therefore, $A(B + C) = AB + AC$

Types of matrices

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- Identity matrix
- The inverse of a matrix
- The transpose of a matrix
- Symmetric matrix
- Orthogonal matrix

Identity matrix

- A square matrix whose elements $a_{ij} = 0$, for $i > j$ is called upper triangular, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

- A square matrix whose elements $a_{ij} = 0$, for $i < j$ is called lower triangular, i.e.,

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Identity matrix

- Both upper and lower triangular, i.e., $a_{ij} = 0$, for $i \neq j$, i.e.,

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

is called a diagonal matrix, simply

$D = \text{diag}[a_{11}, a_{22}, ..., a_{nn}]$

Identity matrix

- In particular, $a_{11} = a_{22} = \ldots = a_{nn} = 1$, the matrix is called identity matrix.
- Properties: $AI = IA = A$

Examples of identity matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The inverse of a matrix

- If matrices A and B such that $AB = BA = I$, then B is called the inverse of A (symbol: A^{-1}); and A is called the inverse of B (symbol: B^{-1}).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Show B is the the inverse of matrix A.

Ans: Note that $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Can you show the details?

The transpose of a matrix

- The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (write A^T).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of A is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

Symmetric matrix

- A matrix A such that $A^T = A$ is called symmetric,

i.e., $a_{ji} = a_{ij}$ for all i and j.
- $A + A^T$ must be symmetric. Why?

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

- A matrix A such that $A^T = -A$ is called skew-symmetric, i.e., $a_{ji} = -a_{ij}$ for all i and j.
- $A - A^T$ must be skew-symmetric. Why?

Orthogonal matrix

- A matrix A is called orthogonal if $AA^T = A^TA = I$,

66 i.e., $A^T = A^{-1}$

Example: prove that $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal.

Since, $A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$. Hence, $AA^T = A^TA = I$.

Can you show the details?

We'll see that orthogonal matrix represents a rotation in fact!

Properties of matrix

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^T = A$ and $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Properties of matrix

- Example: Prove $(AB)^{-1} = B^{-1}A^{-1}$.
- Since $(AB)\,(B^{-1}A^{-1}) = A(B\,B^{-1})A^{-1} = I$ and
- $(B^{-1}A^{-1})\,(AB) = B^{-1}(A^{-1}\,A)B = I$.
- Therefore, $B^{-1}A^{-1}$ is the inverse of matrix AB.

Transpose matrix:

The new matrix obtained by interchanging the rows and columns of the original matrix is called as the transpose of the matrix. If A = [a_{ij}] be an m × n matrix, then the matrix obtained by interchanging the rows and columns of A would be the transpose of A. It is denoted by A′ or (A^T).

In other words, if $A = [a_{ij}]_{m\times n}$, then $A = [a_{ji}]_{n\times m}$.

For example,

If $A = \begin{bmatrix} a & h & d \\ k & b & l \\ g & f & c \end{bmatrix}$. Then $A = \begin{bmatrix} a & k & g \\ h & b & f \\ d & l & c \end{bmatrix}$

Properties:

1) Transpose of Transpose of a Matrix

The transpose of the transpose of a matrix is the matrix itself: $(A^T)^T = A$. Verify that $(A^T)^T = A$.

2) Transpose of a Sum

The transpose of the sum of two matrices is equivalent to the sum of their transposes: $(A + B)^T = A^T + B^T$.

3) Transpose of a Product

The transpose of the product of two matrices is equivalent to the product of their transposes in reversed order: $(AB)^T = B^T A^T$. The same is true for the product of multiple matrices: $(ABC)^T = C^T B^T A^T$.

Question: Suppose $\mathbf{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{5} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{4} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$. Show that $(AB)^T = B^T A^T$

Solution: Here $\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{5} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{4} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

$$= \begin{pmatrix} 2+12+1 & 4+0-1 & 8+9+3 \\ 2+0+3 & 4+0-3 & 8+0+9 \\ 1+8+5 & 2+0-5 & 4+6+15 \end{pmatrix} = \begin{pmatrix} 15 & 3 & 20 \\ 5 & 1 & 17 \\ 14 & -3 & 25 \end{pmatrix}$$

Therefore, $(\mathbf{AB})^{T} = \begin{pmatrix} 15 & 5 & 14 \\ 3 & 1 & -3 \\ 20 & 17 & 25 \end{pmatrix}$

$$\mathbf{B^T A^T} = \begin{pmatrix} 2 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 2 & 0 & -1 \\ 4 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 15 & 5 & 14 \\ 3 & 1 & -3 \\ 20 & 17 & 25 \end{pmatrix}$$

So, $(\mathbf{AB})^{T} = \mathbf{B^T A^T}$ (Showed)

<u>Exercise:</u> If $\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

i. Show that $(A - B)^T = A^T - B^T$
ii. Prove that $(AB)^T = B^T A^T$
iii. Show that $(ABC)^T = C^T B^T A^T$
iv. Find the value of $A^2 + 3B - 4C + 2I$

<u>Solution:</u> (iv) Now, $A^2 = A \cdot A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 4+6+1 & 6+0+2 & 2+9+5 \\ 4+0+3 & 6+0+6 & 2+0+15 \\ 2+4+5 & 3+0+10 & 1+6+25 \end{pmatrix} = \begin{pmatrix} 11 & 8 & 16 \\ 7 & 12 & 17 \\ 11 & 13 & 32 \end{pmatrix}$$

$A^2 + 3B - 4C + 2I$

$$= \begin{pmatrix} 11 & 8 & 16 \\ 7 & 12 & 17 \\ 11 & 13 & 32 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 & 4 \\ 4 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 8 & 16 \\ 7 & 12 & 17 \\ 11 & 13 & 32 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 12 \\ 12 & 0 & 9 \\ 3 & -3 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 16 & 16 \\ 12 & 0 & 4 \\ 8 & -4 & 8 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 11+3-4+2 & 8+6-16+0 & 16+12-16+0 \\ 7+12-12+0 & 12+0-0+2 & 17+9-4+0 \\ 11+3-8+0 & 13-3+4+0 & 32+9-8+2 \end{pmatrix}$$

$$=\begin{pmatrix} 12 & -2 & 12 \\ 7 & 14 & 22 \\ 6 & 14 & 35 \end{pmatrix}$$

Week 13

Topics- Symmetric and skew-symmetric matrix Orthogonal, Idempotent and Involutary Matrix

Page no. (70-75)

Symmetric Matrix:

A square matrix 'A' is said to be symmetric if $A=A^T$, that is, matrix A is said to be symmetric if the transpose of matrix A is equal to matrix A itself.

Skew-Symmetric Matrix:

A square matrix A is said to be skew-symmetric if $a_{ij} = a_{ji}$ ($A= -A^T$) for all i and j. In other words, we can say that matrix A is said to be skew-symmetric if transpose of matrix A is equal to negative of matrix A. Note that all the main diagonal elements in the skew-symmetric matrix are zero.

Let's take an example of a matrix $A = \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & -6 \\ 5 & 6 & 0 \end{pmatrix}$

Then $A^T = \begin{pmatrix} 0 & -3 & 5 \\ 3 & 0 & 6 \\ -5 & -6 & 0 \end{pmatrix}$ and $-A^T = \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & -6 \\ 5 & 6 & 0 \end{pmatrix}$ = A. Therefore, $A = -A^T$

Theorem-1: For any square matrix A with **real number** entries, $\boldsymbol{A + A^T}$ is a symmetric matrix and $\boldsymbol{A - A^T}$ is a skew-symmetric matrix.

Proof: Let $B = A + A^T$,

Then $B^T = (A + A^T)^T$

$= A^T + (A^T)^T$ [as $(A + B)^T = A^T + B^T$]

$= A^T + A$ [$as (A^T)^T = A$]

$= A + A^T = B$

Therefore, B = $A + A^T$ is a symmetric matrix.

Now let $C = A - A^T$

Then $C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$

Therefore, C $= A - A^T$ is a skew-symmetric matrix.

Theorem- 2: Any Square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Proof: Let A be a square matrix then, we can write

$$A = \frac{2A}{2} = \frac{A + A^T + A - A^T}{2}$$

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) = S + K$$

From the Theorem 1, we know that $\left(A + A^T\right)$ is a symmetric matrix and $\left(A - A^T\right)$ is a skew-symmetric matrix. Since for any matrix A, $(kA)^T = k A^T$, it follows that $\frac{1}{2}\left(A + A^T\right)$ is a symmetric matrix and $\frac{1}{2}\left(A - A^T\right)$is a skew-symmetric matrix.

Question-1: Show that the following matrix can be written as the sum of a symmetric and a skew-symmetric matrix A = $\begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix}$.

Solution: Given, $A = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix}$ then, $A^T = \begin{pmatrix} 1 & 5 & 3 \\ 4 & 7 & 0 \\ 5 & -3 & 5 \end{pmatrix}$

Now, the symmetric matrix of A is,

$$S = A + A^T = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 5 & 3 \\ 4 & 7 & 0 \\ 5 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 9 & 8 \\ 9 & 14 & -3 \\ 8 & -3 & 10 \end{pmatrix}$$

Again the skew-symmetric matrix of A is,

$$K = A - A^T = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 5 & 3 \\ 4 & 7 & 0 \\ 5 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Now, $S+K = \begin{pmatrix} 2 & 9 & 8 \\ 9 & 14 & -3 \\ 8 & -3 & 10 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 10 \\ 10 & 14 & -6 \\ 6 & 0 & 10 \end{pmatrix}$

Thus $\frac{1}{2}(S+K) = \begin{pmatrix} 1 & 4 & 5 \\ 5 & 7 & -3 \\ 3 & 0 & 5 \end{pmatrix} = A$, i.e. $A = \frac{1}{2}(S+K)$

So, A can be written as the sum of symmetric and skew-symmetric matrices.

Question-2: Show that the following matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

$$P = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & 2 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} -2 & 7 & 9 \\ 4 & -4 & 0 \\ -1 & 3 & -5 \end{pmatrix}$$

Orthogonal Matrix:

A square matrix with real numbers or values is termed as an orthogonal matrix if its transpose is equal to the inverse matrix of it. In other words, the product of a square orthogonal matrix and its transpose will always give an identity matrix.

Suppose A is the square matrix with real values, of order n × n. Also, let A^T is the transpose matrix of A.

Then according to the definition:

If, $A^T = A^{-1}$ condition is satisfied, then

$$A \cdot A^T = A \cdot A^{-1} = I$$

Where 'I' is the identity matrix of the order n × n. A^{-1} is the inverse of matrix A and 'n' denotes the number of rows and columns. Then we will call A as the orthogonal matrix.

Question-3: Show that the following matrix is the example of an orthogonal matrix

$$P = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

Solution: Given, $P = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}$ then, $P^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

Now, $PP^T = \frac{1}{3} \cdot \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

$$= \frac{1}{9}\begin{pmatrix} 1+4+4 & 2-4+2 & -2-2+4 \\ 2-4+2 & 4+4+1 & -4+2+2 \\ -2-2+4 & -4+2+2 & 4+1+4 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since, $P.P^T = I$, so P is an Orthogonal matrix.

Question-4: Show that the following matrix is the example of an orthogonal matrix.

$$A = \frac{1}{7}\begin{pmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{pmatrix}$$

Involutory matrix:

A Matrix A is said to be Involutory if $A^2 = I$ where, **I** is an Identity matrix.

Question: Show that the following matrix is an Involutory **matrix.**

$$A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

Solution: Given, $A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$

Now $A^2 = A.A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix} . \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since, $A^2 = A.A = I$, so A in an Involutory matrix.

<u>Idempotent</u> **<u>Matrix:</u>** **A matrix A** is said to be <u>Idempotent</u> if $A^2 = A$.

<u>Question:</u> Show that the following matrix is an <u>Idempotent</u> **matrix.**

$$B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$$

<u>Solution:</u> Given, $B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

Now $B^2 = B \cdot B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = B$$

Since, $B^2 = B$, so B in an Idempotent matrix.

Week 14
Topics: Determinant,
Inverse Matrix
Page no (74-90)

Determinant:
Determinant in linear algebra is a useful value to provide the value of a square matrix. We denote the determinant of any matrix A by det(A), or |A|.

Determinant Formula:

1. Let us take a matrix of order **1** ×**1** as: $\boldsymbol{A}=[\mathbf{2}]$

   Then its determinant will be: det(A)=|A| =2

2. Let us take a matrix of 2×2 order as:

   $$A = \begin{bmatrix} p & q \\ c & d \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ -2 & 3 \end{bmatrix} = -12 + 10 = -2$$

3. If the matrix is of 3×3 order :

$$B = \begin{pmatrix} p & q & r \\ a & b & c \\ x & y & z \end{pmatrix}$$

Then its determinant will be: Det(B)=|B|= $\mathrm{p(bz - cy) - q(az - cx) + r(ay - bx)}$

<u>Question-5:</u> Determine the determinant of the matrices:

$$T = \begin{pmatrix} 1 & -2 & -4 \\ 2 & -3 & 3 \\ 1 & 0 & -3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 3 \\ -2 & 5 & -2 \\ 3 & 1 & -6 \end{pmatrix}$$

$$|T| = 1(9 - 0) - (-2)(-6 - 3) + (-4)(0 + 3) = 9 - 18 - 12 = -21$$

$$|\mathrm{A}| = 1(-30 + 2) - 1(12 + 6) + 3(-2 - 15) = -28 - 18 - 51 = -97$$

Determinants

Determinant of order 2

Consider a 2×2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Determinant of A, denoted $|A|$, is a number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of order 2

- easy to remember (for order 2 only)..

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

Determinants

- The following properties are true for determinants of *any* order.

1. If every element of a row (column) is zero, then $|A|=0$

2. $|A^T| = |A|$

determinant of a matrix = that of its transpose

3. $|AB| = |A||B|$

Determinants

- Example: Show that the determinant of any orthogonal matrix is either +1 or -1.
- For any orthogonal matrix, $AA^T = I$.
- Since $|AA^T| = |A||A^T| = 1$ and $|A^T| = |A|$, so $|A|^2 = 1$ or
- $|A| = \pm 1$.

Determinants

For any 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Its inverse can be written as $A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Example: Find the inverse of $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

The determinant of A is -2

Hence, the inverse of A is $A^{-1} = \begin{bmatrix} -1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

How to find an inverse for a 3x3 matrix?

Determinants of order 3

Consider an example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Its determinant can be obtained by:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3(-3) - 6(-6) + 9(-3) = 0$$

You are encouraged to find the determinant by using other rows or columns

Inverse Matrix:

If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}.

For example, let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ be two matrices.

So, $AB = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Similarly $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Thus B is the inverse of A. In other words B = A^{-1} and A is the inverse of B. ie. A = B^{-1}

Question: Find the inverse of the following matrix and show that $A.A^{-1} = A^{-1}.A = I$.

$$A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$$

Solution: Given, $A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

So, $|A| = 1.(0+1) - 4.(6-2) + 4.(-3-0) = 1 - 16 - 12 = -27$

Cofactors:

$A_{11} = (0+1) = 1$

$A_{12} = -(6-2) = -4$

$A_{13} = (-3-0) = -3$

$A_{21} = -(8+4) = -12$

$A_{22} = (2-8) = -6$

$A_{23} = -(-1-8) = 9$

$A_{31} = (4-0) = 4$

$A_{32} = -(1-12) = 11$

$A_{33} = (0-12) = -12$

Now, $Adj(A) = \begin{pmatrix} 1 & -4 & -3 \\ -12 & -6 & 9 \\ 4 & 11 & -12 \end{pmatrix}^T = \begin{pmatrix} 1 & -12 & 4 \\ -4 & -6 & 11 \\ -3 & 9 & -12 \end{pmatrix}$

So, $A^{-1} = \frac{1}{|A|}.Adj(A) = \frac{1}{-27}\begin{pmatrix} 1 & -12 & 4 \\ -4 & -6 & 11 \\ -3 & 9 & -12 \end{pmatrix}$

2nd part: $A.A^{-1} = A \cdot A^{-1} = \frac{1}{-27}\begin{pmatrix} 1 & 4 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -12 & 4 \\ -4 & -6 & 11 \\ -3 & 9 & -12 \end{pmatrix}$

$$= \frac{1}{-27}\begin{pmatrix} 1-16-12 & -12-24+36 & 4+44-48 \\ 3-0-3 & -36-0+9 & 12+0-12 \\ 2+4-6 & -24+6+18 & 8-11-24 \end{pmatrix}$$

$$= \frac{1}{-27}\begin{pmatrix} -27 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Inverse of a matrix

Inverse of a 2 × 2 Matrix

The inverse of a 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If the determinant is 0, is undefined. So a matrix with a determinant of 0 has no inverse. It is called a *singular* matrix.

Example 1: Finding the Inverse of a Matrix

Find the inverse of the matrix if it is defined.

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

First, check that the determinant is nonzero.
4(1) – 2(3) = 4 – 6 = –2. The determinant is –2, so the matrix has an inverse.

The inverse of $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ is $A^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$.

Example 2: Finding the Inverse of a Matrix

Find the inverse of the matrix if it is defined.

$$\begin{bmatrix} 4 & -3 \\ -\frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

The determinant is, $4\left(\frac{1}{4}\right) - \left(-\frac{1}{3}\right)(-3) = 0$, so *B* has no inverse.

Inverse of a 3×3 matrix

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

The cofactor for each element of matrix A:

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

Inverse of a 3×3 matrix

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is then given by:

$$\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Inverse of a 3×3 matrix

Inverse matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

Question: If $P = \begin{pmatrix} 1 & -1 & 4 \\ -4 & -6 & 1 \\ -3 & 1 & -1 \end{pmatrix}$, then find P^{-1} and show that $PP^{-1} = I$

Week 15
Topics: System of Linear Equations , Matrix Method, Cramer's Rule
Page no (90-115)

Solving system of linear equations using matrix method:

Suppose, a system of linear equations are,

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

We can write this as $AX = b$ ……………. (1), where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

From (1) we can write, $X = A^{-1}.b$…………….(2).

Question-1: Solve the following system of linear equations using matrix method.

$$2x - 2y + z = 1$$
$$x + 3y - 2z = 1$$
$$3x - y - z = -2$$

Given system of equations can be written as $AX = b$ ……(2), where

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 3 & -2 \\ 3 & -1 & -1 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

From (2) we can write $X = A^{-1}b$ ………..(3)

Now, $|A| = 2(-3-2) + 2(-1+6) + 1(-1-9) = -10$

Cofactors:

$A_{11} = (-3-2) = -5$

$A_{12} = -(-1+6) = -5$

$A_{13} = (-1-9) = -10$

$A_{21} = -(2+1) = -3$

$A_{22} = (-2-3) = -5$

$A_{23} = -(-2+6) = -4$

$A_{31} = (4-3) = 1$

$A_{32} = -(-4-1) = 5$

$A_{33} = (6+2) = 8$

$$\text{So, } AdjA = \begin{pmatrix} -5 & -5 & -10 \\ -3 & -5 & -4 \\ 1 & 5 & 8 \end{pmatrix}^T = \begin{pmatrix} -5 & -3 & 1 \\ -5 & -5 & 5 \\ -10 & -4 & 8 \end{pmatrix}$$

$$So, A^{-1} = \frac{1}{|A|}.Adj(A) = \frac{1}{-10}\begin{pmatrix} -5 & -3 & 1 \\ -5 & -5 & 5 \\ -10 & -4 & 8 \end{pmatrix}$$

From (3) we get, $X = \frac{1}{-10}\begin{pmatrix} -5 & -3 & 1 \\ -5 & -5 & 5 \\ -10 & -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -5-3-2 \\ -5-5-10 \\ -10-4-16 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -10 \\ -20 \\ -30 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Solution of the system $x = 1,\ y = 2,\ z = 3$

Question-2: Solve the following system of linear equations using matrix method.

$$(i)\quad \begin{aligned} x+2y+z&=3\\ 2x+3y+z&=4\\ 3x-y-z&=4 \end{aligned} \qquad (ii)\quad \begin{aligned} x-2y+3z&=7\\ 2x+y-z&=1\\ x-y-z&=-6 \end{aligned}$$

Cramer's rule:

Suppose, a system of linear equations are,

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

We can write this as $AX = b$ ... ... (1), where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Now $$A_x = \begin{pmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & c_2 & c_3 \end{pmatrix},\quad A_y = \begin{pmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ c_1 & d_3 & c_3 \end{pmatrix},\quad A_z = \begin{pmatrix} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ c_1 & c_2 & d_3 \end{pmatrix}$$

According to Cramer's rule, $x = \frac{|A_x|}{|A|}, y = \frac{|A_y|}{|A|},\ z = \frac{|A_z|}{|A|},$

Cramer's Rule

Gabriel Cramer was a Swiss mathematician (1704-1752)

Introduction

- Cramer's Rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.

- Cramer's Rule relies on determinants

Coefficient Matrices

- You can use determinants to solve a system of linear equations.
- You use the coefficient matrix of the linear system.
- **Linear System**

- ax+by=e
- cx+dy=f

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- **Coeff Matrix**

Cramer's Rule for 2x2 System

- Let A be the coefficient matrix
- <u>**Linear System**</u> <u>**Coeff Matrix**</u>

- ax+by=e
- cx+dy=f

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathrm{ad} - \mathrm{bc}$$

- If det A ≠ 0, then the system has exactly one solution:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

Key Points

- The denominator consists of the coefficients of variables (x in the first column, and y in the second column).
- The numerator is the same as the denominator, with the constants replacing the coefficients of the variable for which you are solving.

Example - Applying Cramer's Rule on a System of Two Equations

Solve the system:

- 8x+5y= 2
- 2x-4y= -10

The coefficient matrix is: $\begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$ $\begin{vmatrix} 8 & 5 \\ 2 & -4 \end{vmatrix} = (-32) - (10) = -42$ and

So: $x = \dfrac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42}$ and $y = \dfrac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42}$

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42} = \frac{-8-(-50)}{-42} = \frac{42}{-42} = -1$$

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42} = \frac{-80-4}{-42} = \frac{-84}{-42} = 2$$

Solution: (-1,2)

Applying Cramer's Rule on a System of Two Equations

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

$$D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D}$$

$$\begin{cases} 2x - 3y = -16 \\ 3x + 5y = 14 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix} = (2)(5) - (-3)(3) = 10 + 9 = 19$$

$$D_x = \begin{vmatrix} -16 & -3 \\ 14 & 5 \end{vmatrix} = (-16)(5) - (-3)(14) = -80 + 42 = -38$$

$$D_y = \begin{vmatrix} 2 & -16 \\ 3 & 14 \end{vmatrix} = (2)(14) - (3)(-16) = 28 + 48 = 76$$

$$x = \frac{D_x}{D} = \frac{-38}{19} = -2 \quad y = \frac{D_y}{D} = \frac{76}{19} = 4$$

Evaluating a 3x3 Determinant
(expanding along the top row)

- Expanding by Minors (little 2x2 determinants)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 3 \\ 1 & 2 & 3 \end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} - (3)\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (1)(-6) \quad - (3)(3) \quad + (-2)(4)$$

$$= -6 \quad -9 \quad -8 = -23$$

Using Cramer's Rule to Solve a System of Three Equations

Consider the following set of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Using Cramer's Rule to Solve a System of Three Equations

The system of equations above can be written in a matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Using Cramer's Rule to Solve a System of Three Equations

Define

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; and \; [B] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $D \neq 0$, then the system has a unique solution as shown below (Cramer's Rule).

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

Using Cramer's Rule to Solve a System of Three Equations

where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{vmatrix} \qquad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{12} & b_2 & a_{23} \\ a_{13} & b_3 & a_{33} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ a_{13} & a_{32} & b_3 \end{vmatrix}$$

Example 1

Consider the following equations:

$$2x_1 - 4x_2 + 5x_3 = 36$$

$$-3x_1 + 5x_2 + 7x_3 = 7$$

$$5x_1 + 3x_2 - 8x_3 = -31$$

$$[A][x] = [B]$$

where

$$[A] = \begin{bmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{bmatrix}$$

Example 1

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad and \quad [B] = \begin{bmatrix} 36 \\ 7 \\ -31 \end{bmatrix}$$

$$D = \begin{vmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{vmatrix} = -336$$

$$D_1 = \begin{vmatrix} 36 & -4 & 5 \\ 7 & 5 & 7 \\ -31 & 3 & -8 \end{vmatrix} = -672$$

Example 1

$$D_2 = \begin{vmatrix} 2 & 36 & 5 \\ -3 & 7 & 7 \\ 5 & -31 & -8 \end{vmatrix} = 1008$$

$$D_3 = \begin{vmatrix} 2 & -4 & 36 \\ -3 & 5 & 7 \\ 5 & 3 & -31 \end{vmatrix} = -1344$$

$$x_1 = \frac{D_1}{D} = \frac{-672}{-336} = 2$$

$$x_2 = \frac{D_2}{D} = \frac{1008}{-336} = -3$$

$$x_3 = \frac{D_3}{D} = \frac{-1344}{-336} = 4$$

Cramer's Rule - 3 x 3

- Consider the 3 equation system below with variables x, y and z:

$$a_1x + b_1y + c_1z = C_1$$
$$a_2x + b_2y + c_2z = C_2$$
$$a_3x + b_3y + c_3z = C_3$$

Cramer's Rule - 3 x 3

- The formulae for the values of x, y and z are shown below. Notice that all three have the same denominator.

$$x = \frac{\begin{vmatrix} C_1 & b_1 & c_1 \\ C_2 & b_2 & c_2 \\ C_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & C_1 & c_1 \\ a_2 & C_2 & c_2 \\ a_3 & C_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & C_1 \\ a_2 & b_2 & C_2 \\ a_3 & b_3 & C_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Example 1

- Solve the system : $3x - 2y + z = 9$
- $x + 2y - 2z = -5$

  $x + y - 4z = -2$

$$x = \frac{\begin{vmatrix} 9 & -2 & 1 \\ -5 & 2 & -2 \\ -2 & 1 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{-23}{-23} = 1 \qquad y = \frac{\begin{vmatrix} 3 & 9 & 1 \\ 1 & -5 & -2 \\ 1 & -2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{69}{-23} = -3$$

Example 1

$$z = \frac{\begin{vmatrix} 3 & -2 & 9 \\ 1 & 2 & -5 \\ 1 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{0}{-23} = 0$$

The solution is

(1, -3, 0)

Cramer's Rule

- Not all systems have a definite solution. If the determinant of the coefficient matrix is zero, a solution cannot be found using Cramer's Rule because of division by zero.
- When the solution cannot be determined, one of two conditions exists:
  - The planes graphed by each equation are parallel and there are no solutions.
  - The three planes share one line (like three pages of a book share the same spine) or represent the same plane, in which case there are infinite solutions.

Question-1: Solve the following system of linear equations using Cramer's rule.

$$(i)\quad \begin{aligned} x+2y+z&=3\\ 2x+3y+z&=3\\ 3x-y-z&=-6 \end{aligned} \qquad (ii)\quad \begin{aligned} x-2y+3z&=7\\ 2x+y-z&=1\\ x-y-z&=-6 \end{aligned}$$

Solution: **(i)**

We can write this as $AX=b\ldots\ldots\ldots\ldots\ldots(1)$, where

$$A=\begin{pmatrix}1&2&1\\2&3&1\\3&-1&-1\end{pmatrix},\ X=\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ and } b=\begin{pmatrix}3\\3\\-6\end{pmatrix}$$

Now $A_x=\begin{pmatrix}3&2&1\\3&3&1\\-6&-1&-1\end{pmatrix},\quad A_y=\begin{pmatrix}1&3&1\\2&3&1\\3&-6&-1\end{pmatrix},\ A_z=\begin{pmatrix}1&2&3\\2&3&3\\3&-1&-6\end{pmatrix}$

$$|A|=\begin{vmatrix}1&2&1\\2&3&1\\3&-1&-1\end{vmatrix}=-3,\quad |A_x|=\begin{vmatrix}3&2&1\\3&3&1\\-6&-1&-1\end{vmatrix}=3,$$

$$|A_y|=\begin{vmatrix}1&3&1\\2&3&1\\3&-6&-1\end{vmatrix}=-3,\quad |A_z|=\begin{vmatrix}1&2&3\\2&3&3\\3&-1&-6\end{vmatrix}=-6$$

According to Cramer's rule, $x=\frac{|A_x|}{|A|}=-1,\ y=\frac{|A_y|}{|A|}=1,\ z=\frac{|A_z|}{|A|}=2$

Week 16
Topics : Rank of a matrix
Page no (115-122)

RANK OF A MATRIX

- The rank of a matrix is equal to the number of linearly independent rows (or columns) in it. Hence, it cannot more than its number of rows and columns. For example, if we consider the identity matrix of order 3 × 3, all its rows (or columns) are linearly independent and hence its rank is 3.

HOW TO FIND THE RANK OF A MARIX?

- The rank of a matrix can be found using three methods. The most easiest of these methods is "converting matrix into echelon form".
- Minor method
- Using echelon form
- Using normal form
- Let us study each of these methods in detail.

Finding Rank of a Matrix by Minor Method

- Here are the steps to find the rank of a matrix A by the minor method.
- Find the determinant of A (if A is a square matrix). If det (A) ≠ 0, then the rank of A = order of A.
- If either det A = 0 (in case of a square matrix) or A is a rectangular matrix, then see whether there exists any minor of maximum possible order is non-zero. If there exists such non-zero minor, then rank of A = order of that particular minor.
- Repeat the above step if all the minors of the order considered in the above step are zeros and then try to find a non-zero minor of order that is 1 less than the order from the above step.
- Here is an example.
- Example: Find the rank of the matrix ρ (A) if A = .
- Solution:
- A is a square matrix and so we can find its determinant.
- det (A) = 1 (45 - 48) - 2 (36 - 42) + 3 (32 - 35)
= -3 + 12 - 9
= 0
- So ρ (A) ≠ order of the matrix. i.e., ρ (A) ≠ 3.
- Now, we will see whether we can find any non-zero minor of order 2.
- = 5 - 8 = -3 ≠ 0.
- So there exists a minor of order 2 (or 2 × 2) which is non-zero. So the rank of A, ρ (A) = 2.

Finding Rank of a Matrix Using Echelon Form

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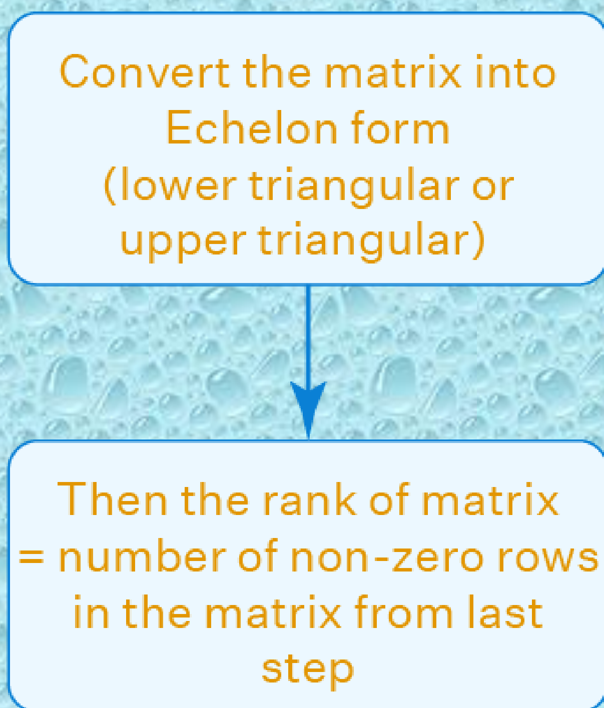

Rank of a Matrix

The maximum number of linearly independent rows in a matrix *A* is called the row rank of *A*, and the maximum number of linearly independent columns in *A* is called the column rank of *A*. If *A* is an *m* by *n* matrix, that is, if *A* has *m* rows and *n* columns, then it is obvious that

$$\left.\begin{matrix} \textbf{Raw rank of } A \leq m \\ \textbf{Column rank of } A \leq n \end{matrix}\right\} \qquad \cdots\cdots (1)$$

However, is that for any matrix *A*, **the row rank of *A* = the column rank of *A***
Because of this fact, there is no reason to distinguish between row rank and column rank; the common value is simply called the **rank** of the matrix. Therefore, if *A* is *m x n*, it follows from the inequalities in (1) that

$$\boldsymbol{rank(A_{m\times n}) \leq min(m, n)} \qquad \cdots\cdots (2)$$

Example-1: Find the rank of the matrix $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$

Solution:

First, because the matrix is 4×3, its rank can be no greater than 3. Therefore, at least one of the four rows will become a row of zeros. Perform the following row operations:

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \xrightarrow[r_4' = -r_1 + r_4]{r_2' = -2r_1 + r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow[r_4' = r_2 + r_4]{r_3' = 2r_2 + r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[r_4' = -2r_3 + r_4]{r_2' = -r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_3' = -r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are 3 nonzero rows remaining in this echelon form of B, rank(B)=3

Example-2: Determine the rank of the following matrices.

$$C = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$$

Question: Express M as a linear combination of the matrices A, B and C where

$$M = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$$

Solution: Given,

$$M = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$$

Here, M will be a linear combination of A, B, C if there exist some scalars a_1, a_2, a_3 such that

$a_1.A + a_2.B + a_3.C = M$ ………………….(1)

Or, $a_1.\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + a_2.\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + a_3.\begin{bmatrix}1 & 1\\4 & 5\end{bmatrix} = \begin{bmatrix}4 & 7\\7 & 9\end{bmatrix}$

Or $\begin{bmatrix}a_1 & a_1\\a_1 & a_1\end{bmatrix} + \begin{bmatrix}a_2 & 2a_2\\3a_2 & 4a_2\end{bmatrix} + \begin{bmatrix}a_3 & a_3\\4a_3 & 5a_3\end{bmatrix} = \begin{bmatrix}4 & 7\\7 & 9\end{bmatrix}$

Or $\begin{bmatrix}a_1 + a_2 + a_3 & a_1 + 2a_2 + a_3\\a_1 + 3a_2 + 4a_3 & a_1 + 4a_2 + 5a_3\end{bmatrix} = \begin{bmatrix}4 & 7\\7 & 9\end{bmatrix}$

$$a_1 + a_2 + a_3 = 4$$

$$a_1 + 2a_2 + a_3 = 7$$

$$a_1 + 3a_2 + 4a_3 = 7$$

$$a_1 + 4a_2 + 5a_3 = 9$$

Solving the above system of equations, we get, $a_1 = 2,\ a_2 = 3,\quad a_3 = -1.$

From (i), we get, $2A + 3B - C = M$ or, $M = 2A + 3B - C$. (Expressed)

Week 17
Topics: Eigenvalue and Eigenvector Cayley Hamilton Theorem

Page no (123-140)

Eigenvalue Definition

Eigenvalues are the special set of scalars associated with the system of linear equations. It is mostly used in matrix equations. 'Eigen' is a German word that means 'proper' or 'characteristic'. Therefore, the term eigenvalue can be termed as characteristic value, characteristic root, proper values or latent roots as well. In simple words, the eigenvalue is a scalar that is used to transform the eigenvector. The basic equation is

Ax = λx

The number or scalar value "**λ**" is an eigenvalue of A.

In Mathematics, an eigenvector corresponds to the real non zero eigenvalues which point in the direction stretched by the transformation whereas eigenvalue is considered as a factor by which it is stretched. In case, if the eigenvalue is negative, the direction of the transformation is negative.

For every real matrix, there is an eigenvalue. Sometimes it might be complex. The existence of the eigenvalue for the complex matrices is equal to the fundamental theorem of algebra.

What are Eigenvectors?

Eigenvectors are the vectors (non-zero) that do not change the direction when any linear transformation is applied. It changes by only a scalar factor. In a brief, we can say, if A is a linear transformation from a vector space V and **x** is a vector in V, which is not a zero vector, then v is an eigenvector of A if A(X) is a scalar multiple of **x**.

Before we start

Terminology

If s is a non-zero vector such that $\mathbf{M}s = \lambda s$

s is called an **eigenvector** of **M**; λ is the corresponding **eigenvalue**.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is an eigenvector of } \mathbf{M} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \text{ because } \mathbf{M}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Before we start

<u>Properties of eigenvectors</u>

1. For a given eigenvalue, the eigenvector is not unique.

   E.g. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{M} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ because $\mathbf{M}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   But also $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

2. Under the transformation, the eigenvector is enlarged by a scale factor equal to its eigenvalue.
3. The transformation does not change the direction of the eigenvector.
4. $\lambda = 0$ is a possible eigenvalue.

Before we start…

<u>Finding eigenvectors</u>

We need to solve the equation $\mathbf{Ms} = \lambda\mathbf{s}$

to find $\mathbf{s}$.

But $\mathbf{Ms} = \lambda\mathbf{s}$

$\Leftrightarrow \quad \mathbf{Ms} - \lambda\mathbf{s} = \mathbf{0}$

$\Leftrightarrow \quad \mathbf{Ms} - \lambda\mathbf{Is} = \mathbf{0}$

$\Leftrightarrow \quad (\mathbf{M} - \lambda\mathbf{I})\mathbf{s} = \mathbf{0}$

If $\mathbf{M} - \lambda\mathbf{I}$ is non-singular, this equation has a solution

$$\mathbf{s} = (\mathbf{M} - \lambda\mathbf{I})^{-1}\mathbf{0} = \mathbf{0}$$

but we are seeking non-zero solutions for $\mathbf{s}$.

For these to exist, we require $\mathbf{M} - \lambda\mathbf{I}$ to be singular.

$\Leftrightarrow \quad \det(\mathbf{M} - \lambda\mathbf{I}) = 0$

This is called the **characteristic equation** of $\mathbf{M}$.

Eigenvalues and Eigenvectors of a 2×2 matrix

Example

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

characteristic equation of **M**

$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

For each eigenvalue λ, find a corresponding eigenvector **s** by solving (**M** – λ**I**)**s** = **0** for non-zero **s**.

$$\mathbf{M} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}$$

λ = 2

λ = 3

Eigenvalues and Eigenvectors of 3×3 matrices

The theory is exactly the same for 3 × 3 matrices.

<u>Example</u>

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 6 \\ 3 & -3 & 27 \\ 1 & -1 & 7 \end{bmatrix}$.

Form the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

$$\mathbf{M} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & -1 & 6 \\ 3 & -3-\lambda & 27 \\ 1 & -1 & 7-\lambda \end{bmatrix}$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (2-\lambda)\begin{vmatrix} -3-\lambda & 27 \\ -1 & 7-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & 27 \\ 1 & 7-\lambda \end{vmatrix} + 6\begin{vmatrix} 3 & -3-\lambda \\ 1 & -1 \end{vmatrix} = 0$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (2-\lambda)\begin{vmatrix} -3-\lambda & 27 \\ -1 & 7-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & 27 \\ 1 & 7-\lambda \end{vmatrix} + 6\begin{vmatrix} 3 & -3-\lambda \\ 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)(\lambda^2 - 4\lambda + 6) + 1(-3\lambda - 6) + 6\lambda = 0$$
$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$
$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Solve the characteristic equation.

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

eigenvalues are 1, 2 and 3.

For each eigenvalue λ, find a corresponding eigenvector **s** by solving $(\mathbf{M} - \lambda\mathbf{I})\mathbf{s} = \mathbf{0}$ for non-zero **s**.

$\lambda = 1$

$(\mathbf{M} - \lambda\mathbf{I})\mathbf{s} = \mathbf{0}$

$$\begin{bmatrix} 1 & -1 & 6 \\ 3 & -4 & 27 \\ 1 & -1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ (1) $x - y + 6z = 0$

(2) $3x - 4y + 27z = 0$

(3) $x - y + 6z = 0$

$\Rightarrow$ (1) = (3)

(1) × 3 $3x - 3y + 18z = 0$

– (2): $y - 9z = 0 \Rightarrow y = 9z$

$\Rightarrow$ in (1); $x - 9z + 6z = 0 \Rightarrow x = 3z$

Let $z = 1$ (why not?): $x = 3$ and $y = 9$, so an eigenvector is $\begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$

$\lambda = 2$: an eigenvector is $\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$

$\lambda = 3$: an eigenvector is $\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$

For a 3×3 matrix, the characteristic equation is a cubic. This will always have at least one real root so there will always be at least one real eigenvalue and associated invariant line.

The situation where we do not have three real distinct eigenvalues is beyond the specification, but is explored in the example on p.109.

(useful to remember)

Sum of roots = Tr(**M**)
Product of roots = det(**M**)

For a 2 × 2 matrix $\mathbf{M} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, the characteristic equation is

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0 \Rightarrow (a - \lambda)(d - \lambda) - bc = 0$$

$$\Rightarrow \quad \lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\Rightarrow \quad \lambda^2 - \mathrm{Tr}(\mathbf{M})\lambda + \det(\mathbf{M}) = 0$$

These properties also hold for $n \times n$ matrices.

An **Eigenspace** of vector **x** consists of a set of all eigenvectors with the equivalent eigenvalue collectively with the zero vector. Though, the zero vector is not an eigenvector.

Let us say A is an "n × n" matrix and λ is an eigenvalue of matrix A, then **x**, a non-zero vector, is called as eigenvector if it satisfies the given below expression;

$A\mathbf{x} = \lambda\mathbf{x}$

x is an eigenvector of A corresponding to eigenvalue, λ.

Note:

- There could be infinitely many Eigenvectors, corresponding to one eigenvalue.
- For distinct eigenvalues, the eigenvectors are linearly dependent.

Eigenvalues of a Square Matrix

Suppose, $A_{n\times n}$ is a square matrix, then [A- λI] is called an Eigen or characteristic matrix, which is an indefinite or undefined scalar. Where determinant of Eigen matrix can be written as, **|A- λI| and |A- λI| = 0** is the Eigen equation or characteristics equation, where "I" is the identity matrix. The roots of an Eigen matrix are called Eigen roots.

Eigenvalues of a triangular matrix and diagonal matrix are equivalent to the elements on the principal diagonals. But eigenvalues of the scalar matrix are the scalar only.

Question: Find the eigenvalue of the given matrix $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$

Solution: The characteristic equation of A is $|\lambda I - A| = 0$

$$\Rightarrow \left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 4 & -6 & -6 \\ -1 & \lambda - 3 & -2 \\ 1 & 4 & \lambda + 3 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 4)\left(\lambda^2 - 9 + 8\right) + 6(-\lambda - 3 + 2) - 6(-4 - \lambda + 3) = 0$$

$$\Rightarrow (\lambda - 4)\left(\lambda^2 - 1\right) - 6\lambda - 6 + 6\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 4)\left(\lambda^2 - 1\right) = 0$$

$$\Rightarrow (\lambda - 4) = 0, \quad \left(\lambda^2 - 1\right) = 0$$

$$\Rightarrow \lambda = 4, \quad \lambda = \pm\sqrt{1} = \pm 1$$

$$\therefore \lambda = 4, -1, 1$$

Therefore, the eigenvalues of A are $\lambda = -1, 1, 4$

Exercise: Find the eigenvalue of the given matrix $B = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Question: Find the characteristic equation of the following matrices and verify Cayley-Hamilton theorem for it. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

<u>Solution:</u> The characteristic equation of A is $|\lambda I - A| = 0$

$$\Rightarrow\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&2&3\\2&-1&1\\3&1&1\end{pmatrix}\right|=0$$

$$\Rightarrow\begin{vmatrix}\lambda-1&-2&-3\\-2&\lambda+1&-1\\-3&-1&\lambda-1\end{vmatrix}=0$$

$$\Rightarrow(\lambda-1)\left(\lambda^2-1-1\right)+2(-2\lambda+2-3)-3(2+3\lambda+3)=0$$

$$\Rightarrow(\lambda-1)\left(\lambda^2-2\right)-4\lambda-2-9\lambda-15=0$$

$$\Rightarrow\lambda^3-2\lambda-\lambda^2+2-13\lambda-17=0$$

$$\therefore\quad\lambda^3-\lambda^2-15\lambda-15=0$$

Now in order to verify Cayley-Hamilton theorem we have to show that
$A^3-A^2-15A-15I=0$

Here, $A=\begin{pmatrix}1&2&3\\2&-1&1\\3&1&1\end{pmatrix}$ $A^2=\begin{pmatrix}1&2&3\\2&-1&1\\3&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&-1&1\\3&1&1\end{pmatrix}=\begin{pmatrix}14&3&8\\3&6&6\\8&6&11\end{pmatrix}$ and

$$A^3=A\cdot A^2=\begin{pmatrix}1&2&3\\2&-1&1\\3&1&1\end{pmatrix}\begin{pmatrix}14&3&8\\3&6&6\\8&6&11\end{pmatrix}=\begin{pmatrix}44&33&53\\33&6&21\\53&21&41\end{pmatrix}$$

$A^3-A^2-15A-15I$

$$=\begin{pmatrix}44&33&53\\33&6&21\\53&21&41\end{pmatrix}-\begin{pmatrix}14&3&8\\3&6&6\\8&6&11\end{pmatrix}-15\begin{pmatrix}1&2&3\\2&-1&1\\3&1&1\end{pmatrix}-15\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}44-14-15-15&33-3-30-0&53-8-45-0\\33-3-30-0&6-6+15-15&21-6-15-0\\53-8-45-0&21-6-15-0&41-11-15-15\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Hence the Cayley-Hamilton theorem is verified.

1. Find Inverse of matrix using Cayley Hamilton method

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution:
To apply the Cayley-Hamilton theorem, we first determine the characteristic polynomial p(t) of the matrix A.
$|A - tI|$

$$= \begin{vmatrix} (3 - t) & 1 & 1 \\ -1 & (2 - t) & 1 \\ 1 & 1 & (1 - t) \end{vmatrix}$$

$$= (3 - t)((2 - t) \times (1 - t) - 1 \times 1) - 1((-1) \times (1 - t) - 1 \times 1) + 1((-1) \times 1 - (2 - t) \times 1)$$

$$= (3 - t)\left(\left(2 - 3t + t^2\right) - 1\right) - 1((-1 + t) - 1) + 1((-1) - (2 - t))$$

$$= (3 - t)\left(1 - 3t + t^2\right) - 1(-2 + t) + 1(-3 + t)$$

$$= \left(3 - 10t + 6t^2 - t^3\right) - (-2 + t) + (-3 + t)$$

$$= -t^3 + 6t^2 - 10t + 2$$

$$p(t) = -t^3 + 6t^2 - 10t + 2$$

The Cayley-Hamilton theorem yields that
$O = p(A) = -A^3 + 6A^2 - 10A + 2I$

Rearranging terms, we have
$\therefore 2I = A^3 - 6A^2 + 10A$

$$\therefore 2I = A\left(A^2 - 6A + 10I\right)$$

$$\therefore A^{-1} = \frac{1}{2}\left(A^2 - 6A + 10I\right)$$

Now, first we find $A^2 - 6A + 10I$

$$A^2 = A \times A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 3 + 1 \times -1 + 1 \times 1 & 3 \times 1 + 1 \times 2 + 1 \times 1 & 3 \times 1 + 1 \times 1 + 1 \times 1 \\ -1 \times 3 + 2 \times -1 + 1 \times 1 & -1 \times 1 + 2 \times 2 + 1 \times 1 & -1 \times 1 + 2 \times 1 + 1 \times 1 \\ 1 \times 3 + 1 \times -1 + 1 \times 1 & 1 \times 1 + 1 \times 2 + 1 \times 1 & 1 \times 1 + 1 \times 1 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 1 + 1 & 3 + 2 + 1 & 3 + 1 + 1 \\ -3 - 2 + 1 & -1 + 4 + 1 & -1 + 2 + 1 \\ 3 - 1 + 1 & 1 + 2 + 1 & 1 + 1 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 5 \\ -4 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 9 & 6 & 5 \\ -4 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix}$$

$$6 \times A = 6 \times \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 6 \\ -6 & 12 & 6 \\ 6 & 6 & 6 \end{bmatrix}$$

$$A^2 - 6 \times A = \begin{bmatrix} 9 & 6 & 5 \\ -4 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix} - \begin{bmatrix} 18 & 6 & 6 \\ -6 & 12 & 6 \\ 6 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 9+18 & 6+6 & 5+6 \\ -4-6 & 4+12 & 2+6 \\ 3+6 & 4+6 & 3+6 \end{bmatrix} = \begin{bmatrix} -9 & 0 & -1 \\ 2 & -8 & -4 \\ -3 & -2 & -3 \end{bmatrix}$$

$$10 \times I = 10 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$A^2 - 6 \times A + 10 \times I = \begin{bmatrix} -9 & 0 & -1 \\ 2 & -8 & -4 \\ -3 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} -9+10 & 0+0 & -1+0 \\ 2+0 & -8+10 & -4+0 \\ -3+0 & -2+0 & -3+10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -4 \\ -3 & -2 & 7 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{2}\left(A^2 - 6A + 10I\right)$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -4 \\ -3 & -2 & 7 \end{bmatrix}$$

Exercise: Find the characteristic equation of the following matrices and verify Cayley-Hamilton theorem for it.

$$B=\begin{pmatrix}1&2&2\\3&1&0\\1&1&1\end{pmatrix}, C=\begin{pmatrix}2&1&1\\1&4&3\\-1&-1&0\end{pmatrix}$$