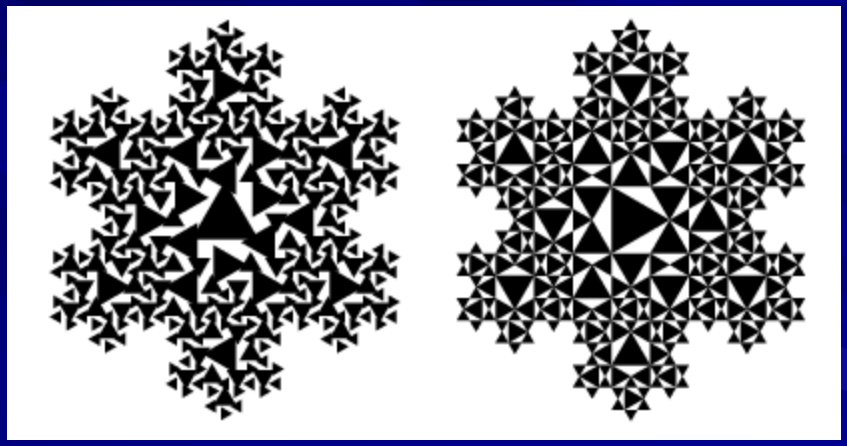
Fundamentals of Fractals



What are Fractals?

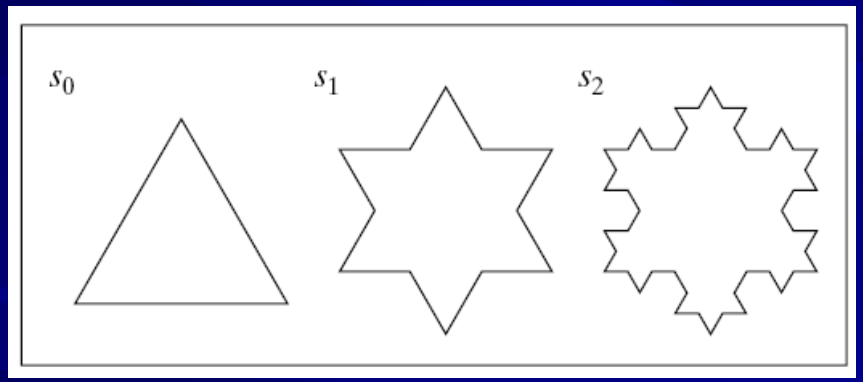
- Fractal definition from MathWorld
  - A *fractal* is an object or quantity that displays self-similarity, in a somewhat technical sense, on all scales.
  - Fractals need not exhibit *exactly* the same structure at all scales, but the same "type" of structures must appear on all scales.

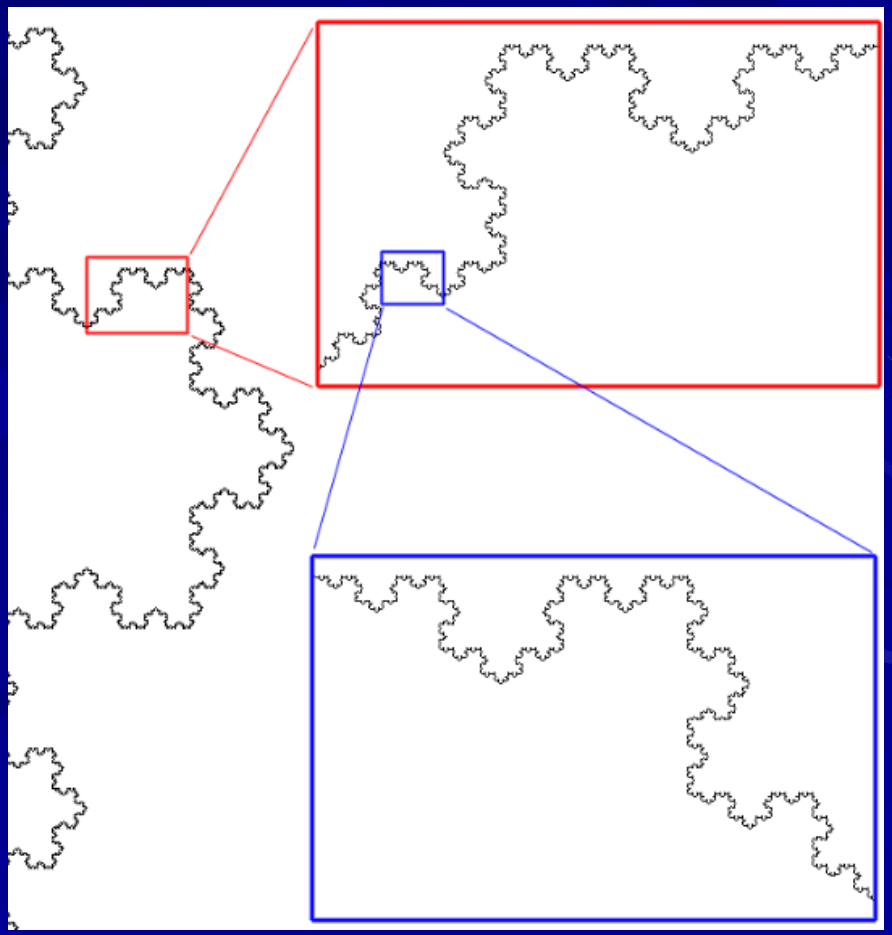
Self-Similarity Property of Fractal

- Self similarity across scales
  - As one zooms in or out the geometry/image has a similar (sometimes exact) appearance
  - Types of self-similarity
    - Exact self similarity
    - Approximate self similarity
    - Statistical self similarity

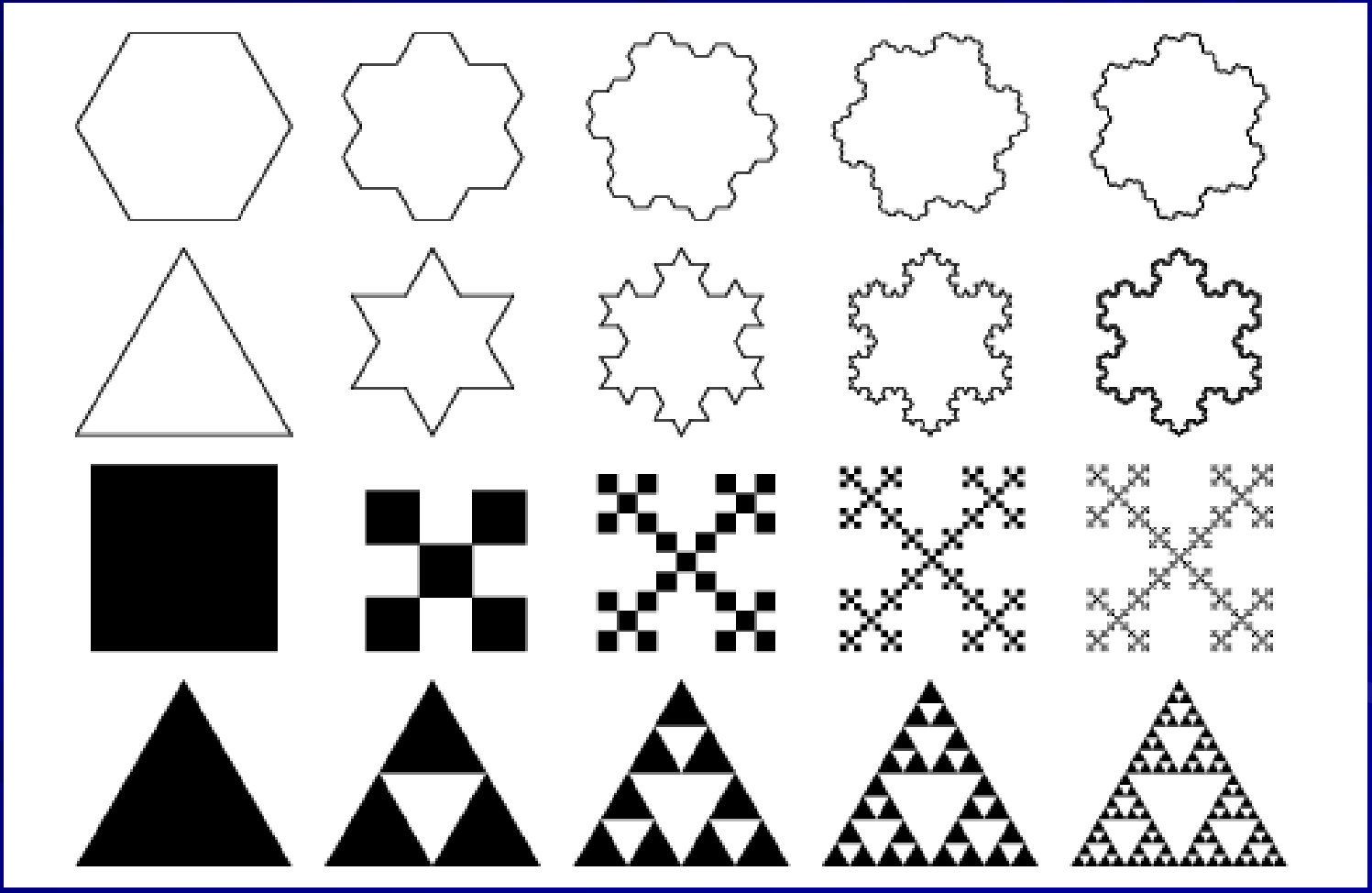
Exact Self-Similarity

- Self similarity may be exact
  - Normally only occurs in mathematically defined fractals
  - Example:
    Koch snowflake



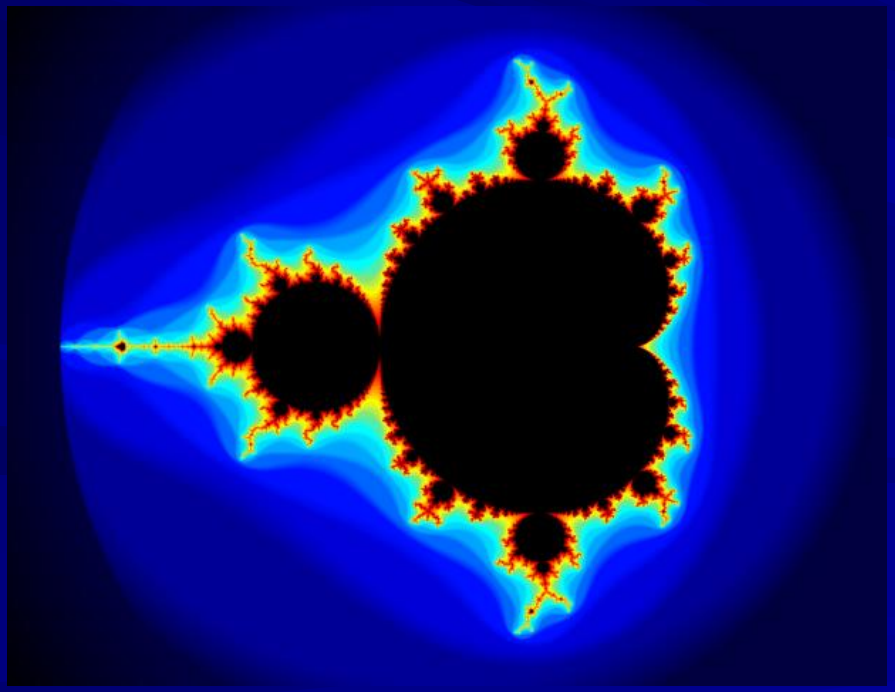


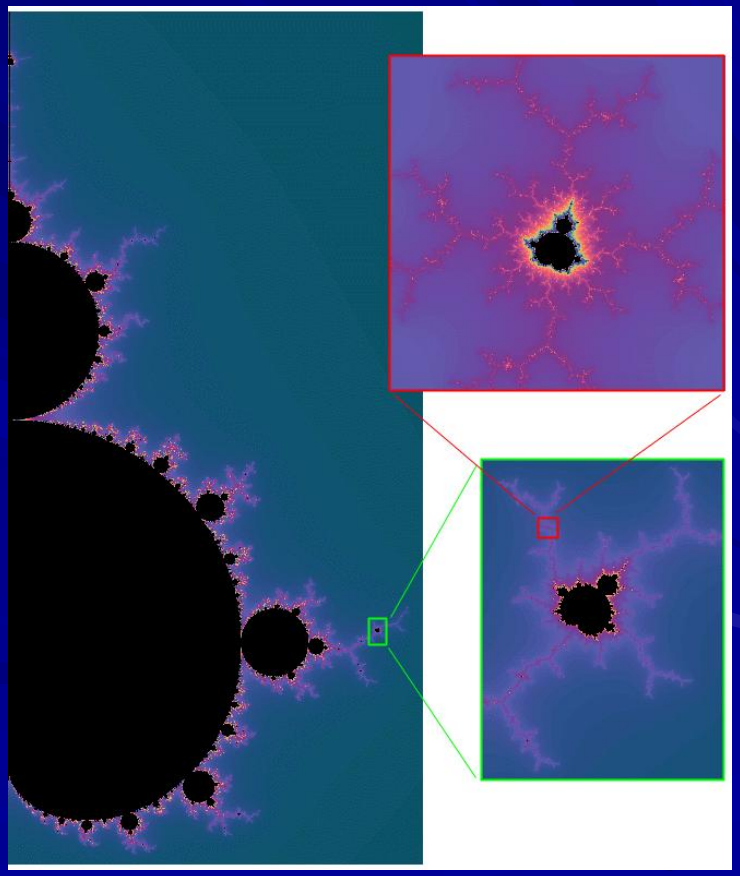
Exact Self-Similarity



Approximate Self-Similarity

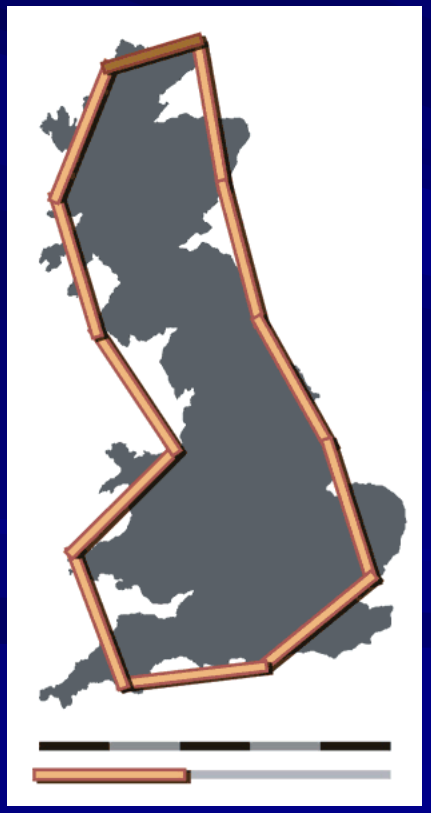
- Structures that are recognizably similar but not exactly so
  - More common type of self-similarity
  - Example: Mandelbrot set

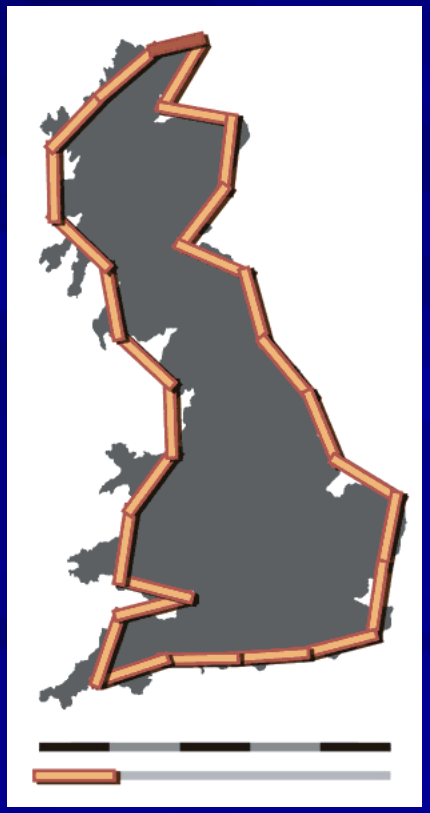




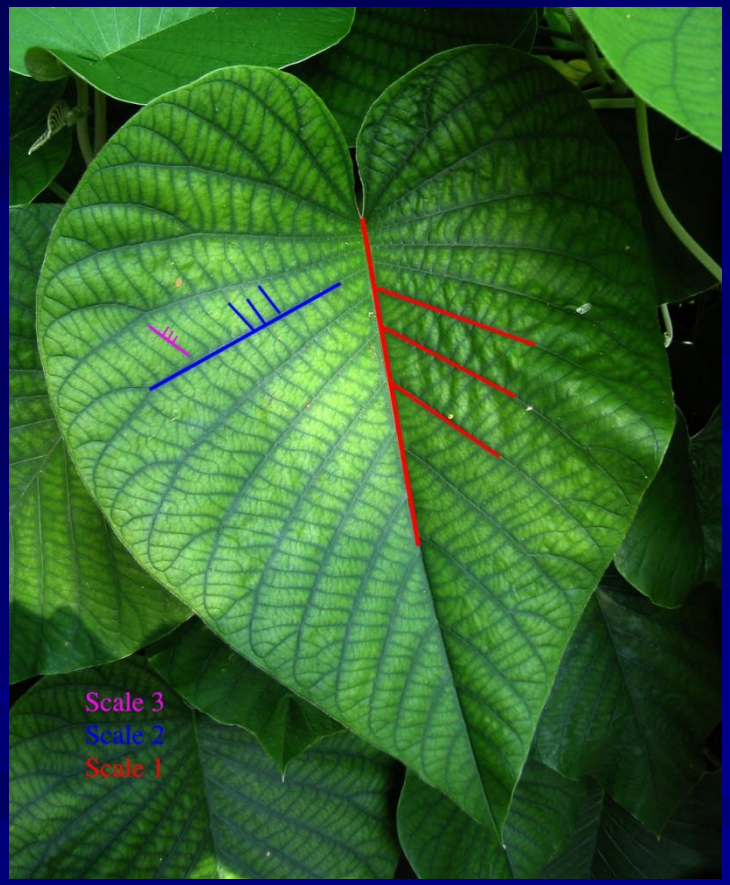
Statistical Self-Similarity

- Irregularity is the same on the average
  - Example: coastline





Self-Similarity in Real World

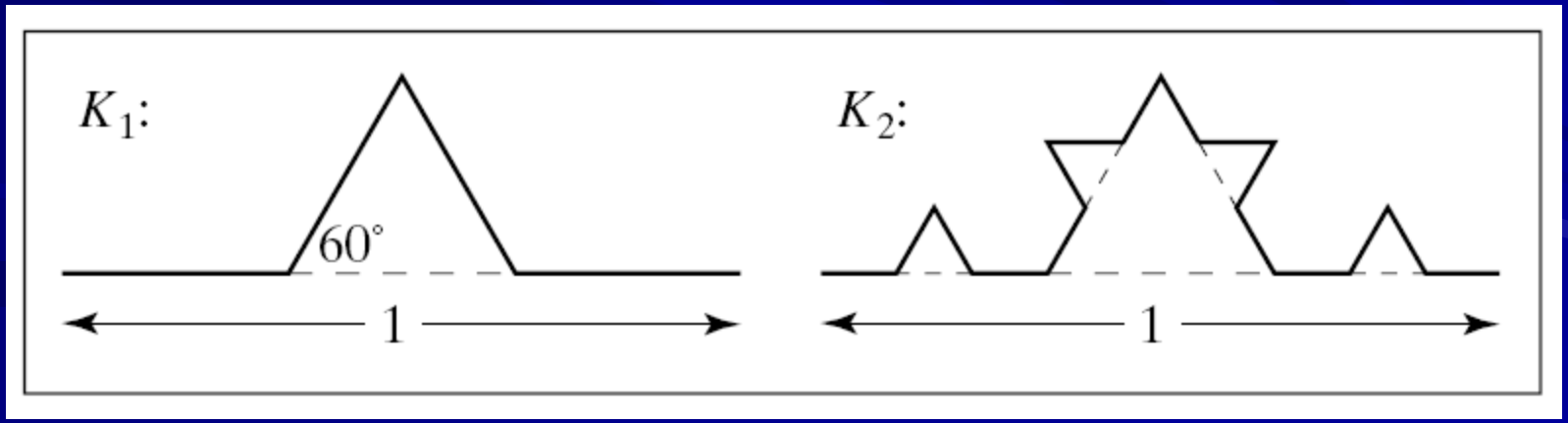


Successive Refinement of Curves

- Very complex curves can be made by repeatedly refining a simple curve.
- The simplest example: Koch curve
  - Discovered in 1904 by Helge von Koch, an Swedish mathematician
  - Exactly self-similar fractal
  - Fascinating feature: an infinitely long line within a region of finite area

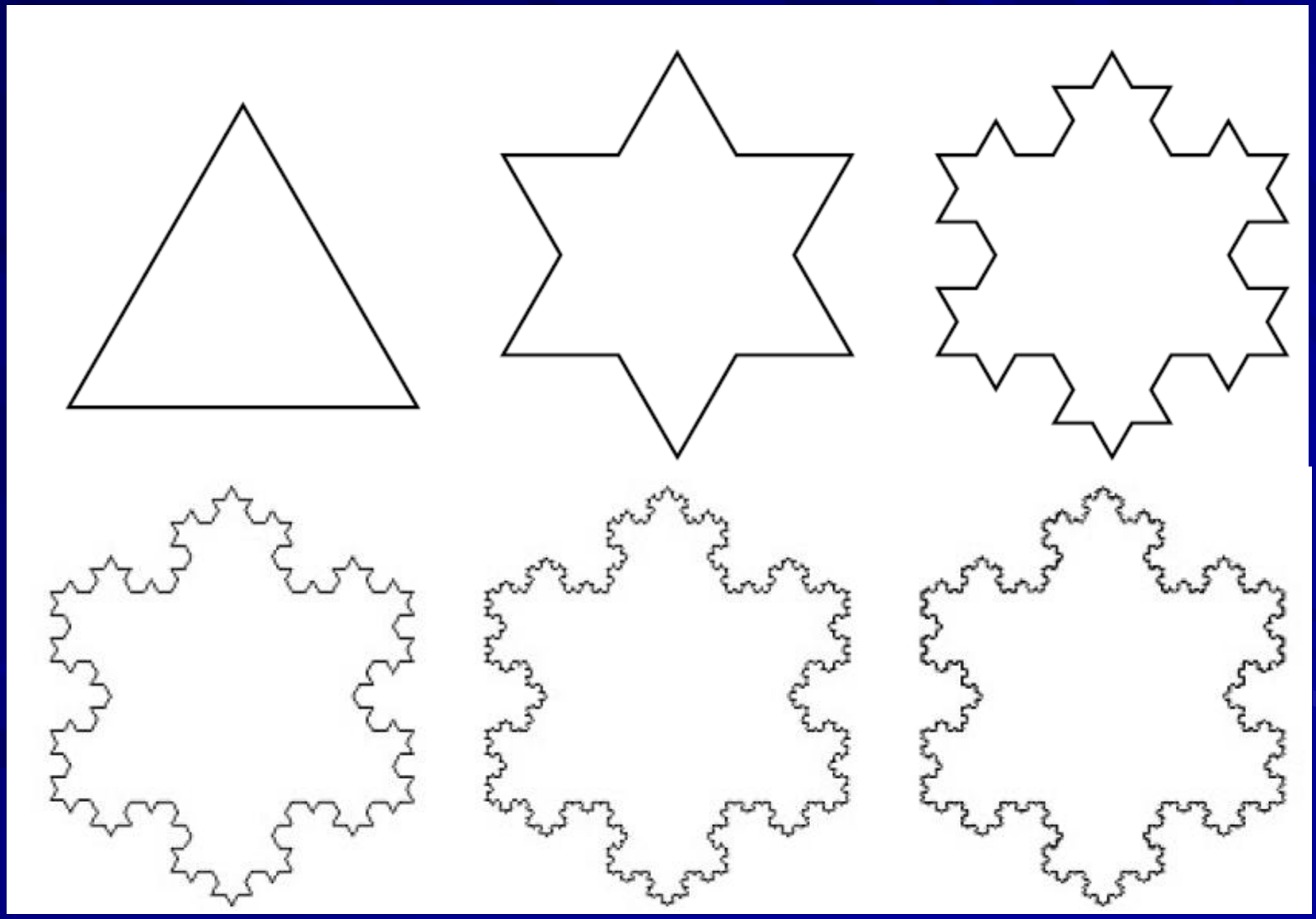
Koch Curves

- K_0, K_1, K_2, …: successive generations of the Koch curve
- K_0: a horizontal line of length unity.
- $K_n \rightarrow K_{n+1}$: subdivide each segment of K_n into three equal parts and replace the middle part with a bump in the shape of an equilateral triangle.



Two generations of the Koch curve

Koch Snowflake

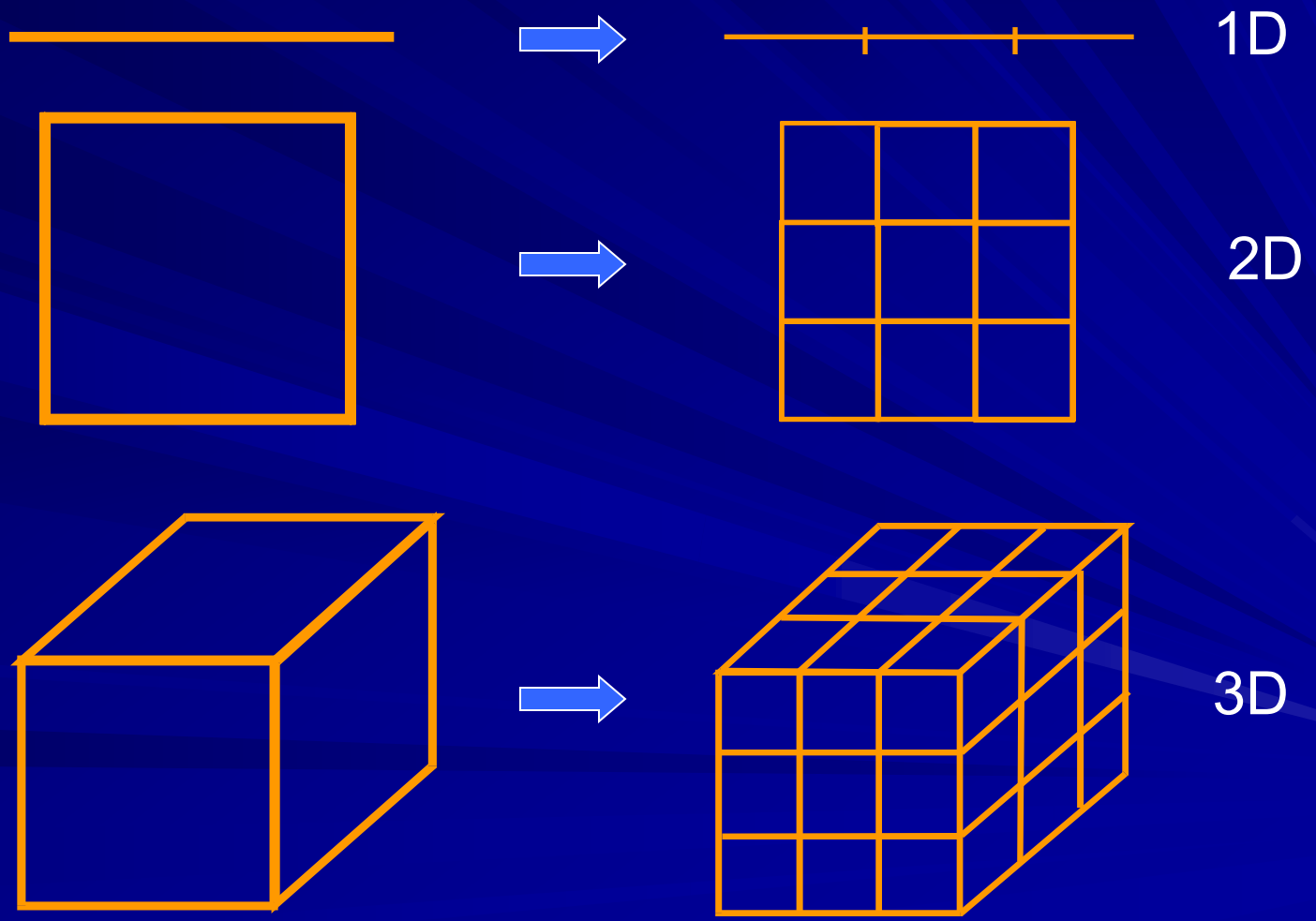


Dimension of Koch Curves?

- A line is *one dimensional* and a plane is *two dimensional*, are there “creatures” in between?
- Koch curves are infinite in length $(4/3)^n$, yet lie inside a finite rectangle
  - Their dimension lies somewhere between 1 and 2.

Fractional Dimension

- Self-similar dimension



Fractional Dimension

- Self-similar dimension
  - An object has dimension *D* if, when it is subdivided into *N* equal parts, each part must be made smaller on each side by $r = 1 / N^{1/D}$.
  - Fractional dimension: $D = \log(N) / \log(1 / r)$
  - Fractional dimension of Koch curve
    - From one generation to the next, $N = 4$ segments are created from each parent segment, but their lengths are $r = 1/3$
    - $D = \log(4) / \log(3) = 1.26$

Drawing Koch Curves

- Another view of Koch curves
  - Each generation consists of four versions of the previous generation.
- Pseudo-code of drawing Koch curve recursively

```
To draw Kn:
  if (n equals 0) Draw a straight line;
  else {
    Draw Kn-1;
    Perform a transformation;
    Draw Kn-1;
    Perform a transformation;
    Draw Kn-1;
    Perform a transformation;
    Draw Kn-1;
  }
```