Computer Graphics

Representing Curves and Surfaces

Amartya Kundu Durjoy
Lecturer, CSE, UGV

Review

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}$$

$$0 \le t \le 1$$

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

Review

$$
\begin{aligned}
Q(t) &= \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} \\
&= T \bullet C \\
&= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}
\end{aligned}
$$

Review

$$\frac{d}{dt}Q(t) = Q'(t) = \begin{bmatrix} \frac{d}{dt}x(t) & \frac{d}{dt}y(t) & \frac{d}{dt}z(t) \end{bmatrix}$$

$$= \begin{bmatrix} x'(t) & y'(t) & z'(t) \end{bmatrix}$$

$$x'(t) = 3a_x t^2 + 2b_x t + c_x$$

$$y'(t) = 3a_y t^2 + 2b_y t + c_y$$

$$z'(t) = 3a_z t^2 + 2b_z t + c_z$$

Review

$$\frac{d}{dt}Q(t) = Q'(t) = \begin{bmatrix} \frac{d}{dt}x(t) & \frac{d}{dt}y(t) & \frac{d}{dt}z(t) \end{bmatrix}$$

$$= \frac{d}{dt}T \bullet C = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \bullet C$$

$$= \begin{bmatrix} 3a_x t^2 + 2b_x t + c_x & 3a_y t^2 + 2b_y t + c_y & 3a_z t^2 + 2b_z t + c_z \end{bmatrix}$$

Review

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T \bullet M \bullet G$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \bullet \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}$$

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \\ g_{3x} & g_{3y} & g_{3z} \\ g_{4x} & g_{4y} & g_{4z} \end{bmatrix}$$

Review

$$
\begin{aligned}
x(t) = {} & (m_{11}t^3 + m_{21}t^2 + m_{31}t + m_{41}) \bullet g_{1x} + \\
& (m_{12}t^3 + m_{22}t^2 + m_{32}t + m_{42}) \bullet g_{2x} + \\
& (m_{13}t^3 + m_{23}t^2 + m_{33}t + m_{43}) \bullet g_{3x} + \\
& (m_{14}t^3 + m_{24}t^2 + m_{34}t + m_{44}) \bullet g_{4x} \\
y(t) = {} & (m_{11}t^3 + m_{21}t^2 + m_{31}t + m_{41}) \bullet g_{1y} + \\
& (m_{12}t^3 + m_{22}t^2 + m_{32}t + m_{42}) \bullet g_{2y} + \\
& (m_{13}t^3 + m_{23}t^2 + m_{33}t + m_{43}) \bullet g_{3y} + \\
& (m_{14}t^3 + m_{24}t^2 + m_{34}t + m_{44}) \bullet g_{4y} \\
z(t) = {} & (m_{11}t^3 + m_{21}t^2 + m_{31}t + m_{41}) \bullet g_{1z} + \\
& (m_{12}t^3 + m_{22}t^2 + m_{32}t + m_{42}) \bullet g_{2z} + \\
& (m_{13}t^3 + m_{23}t^2 + m_{33}t + m_{43}) \bullet g_{3z} + \\
& (m_{14}t^3 + m_{24}t^2 + m_{34}t + m_{44}) \bullet g_{4z}
\end{aligned}
$$

Review

- Blending function
  (also called ‘Basis’ function)

$$B = T \bullet M$$

Hermite Curves

- Find the equation of the curve based on the curve endpoints P1, P4 and the endpoint slope R1, R4.

Hermite Curves

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T \bullet M \bullet G$$

$$G_H = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix}$$

Hermite Curves

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T \bullet M_H \bullet G_H$$

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$= T \bullet M_H \bullet G_{H_x} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet M_H \bullet G_{H_x}$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$
$$= T \bullet M_H \bullet G_{H_y} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet M_H \bullet G_{H_y}$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$
$$= T \bullet M_H \bullet G_{H_z} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet M_H \bullet G_{H_z}$$

Hermite Curves

$$x(0) = P_{1x} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \bullet M_H \bullet G_{H_x}$$

$$y(0) = P_{1y} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \bullet M_H \bullet G_{H_y}$$

$$z(0) = P_{1z} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \bullet M_H \bullet G_{H_z}$$

$$Q(0) = P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \bullet M_H \bullet G_H$$

Hermite Curves

$$x(1) = P_{4x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \bullet M_H \bullet G_{H_x}$$

$$y(1) = P_{4y} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \bullet M_H \bullet G_{H_y}$$

$$z(1) = P_{4z} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \bullet M_H \bullet G_{H_z}$$

$$Q(1) = P_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \bullet M_H \bullet G_H$$

Hermite Curves

$$\frac{d}{dt}Q(t) = Q'(t) = \begin{bmatrix} x'(t) & y'(t) & z'(t) \end{bmatrix} = \frac{d}{dt}T \bullet M_H \bullet G_H$$

$$x'(t) = 3a_x t^2 + 2b_x t + c_x$$

$$= \frac{d}{dt}T \bullet M_H \bullet G_{H_x} = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_x}$$

$$y'(t) = 3a_y t^2 + 2b_y t + c_y$$

$$= \frac{d}{dt}T \bullet M_H \bullet G_{H_y} = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_y}$$

$$z'(t) = 3a_z t^2 + 2b_z t + c_z$$

$$= \frac{d}{dt}T \bullet M_H \bullet G_{H_z} = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_z}$$

Hermite Curves

$$x'(0) = R_{1x} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_x}$$

$$y'(0) = R_{1y} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_y}$$

$$z'(0) = R_{1z} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_z}$$

$$Q'(0) = R_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_H$$

Hermite Curves

$$x'(1) = R_{4x} = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_x}$$

$$y'(1) = R_{4y} = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_y}$$

$$z'(1) = R_{4z} = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_{H_z}$$

$$Q'(1) = R_4 = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_H$$

Hermite Curves

$$\begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = G_H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \bullet M_H \bullet G_H$$

Hermite Curves

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hermite Curves

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T \bullet M_H \bullet G_H$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix}$$

Hermite Curves

$$Q(t) = T \bullet M_H \bullet G_H = B_H \bullet G_H$$

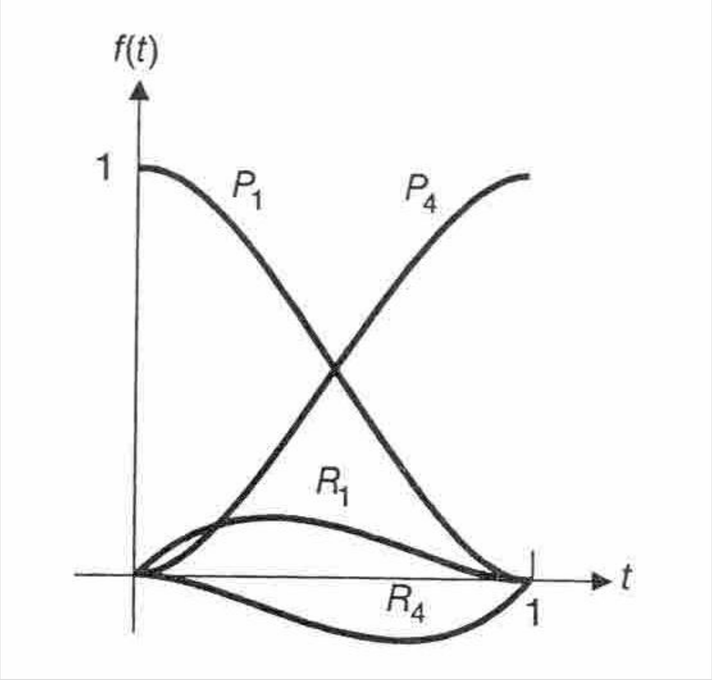
$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet G_H$$

$$= \begin{bmatrix} 2t^3 - 3t^2 + 1 & -2t^3 + 3t^2 & t^3 - 2t^2 + t & t^3 - t^2 \end{bmatrix} \bullet \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}$$

$$= (2t^3 - 3t^2 + 1)P_1 + (-2t^3 + 3t^2)P_4 + (t^3 - 2t^2 + t)R_1 + (t^3 - t^2)R_4$$

Hermite Curve

$$B_H = \left[2t^3 - 3t^2 + 1 \quad -2t^3 + 3t^2 \quad t^3 - 2t^2 + t \quad t^3 - t^2 \right]$$



Hermite Curve

$$G_{H1} = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}, G_{H2} = \begin{bmatrix} P_4 \\ P_7 \\ kR_4 \\ R_7 \end{bmatrix}$$

$$G^1 \ continuity, k > 0$$

$$C^1 \ continuity, k = 1$$

Hermite Curve

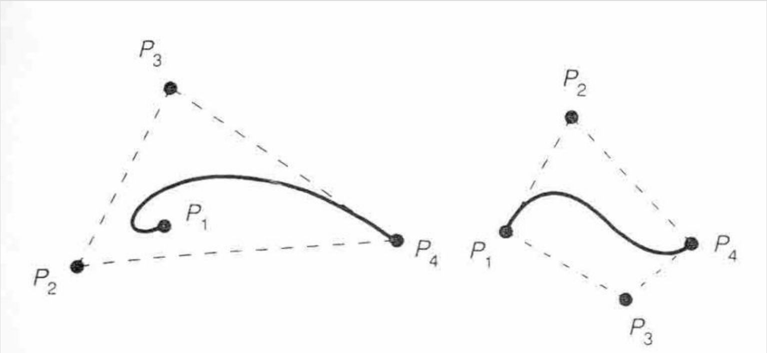
- Reduce 6 multiplies and 3 additions to 3 multiplies and 3 additions.

$$\begin{aligned} f(t) &= at^3 + bt^2 + ct + d \\ &= ((at + b)t + c)t + d \end{aligned}$$

Bezier Curves

- Find the equation of the curve using the curve endpoints P1, P4 and the control points P2, P3
- The slope of the curve endpoint is

$$R_1 = Q'(0) = 3(P_2 - P_1)$$
$$R_4 = Q'(1) = 3(P_4 - P_3)$$



Bezier Curves

$$R_1 = \partial(P_2 - P_1)$$
$$R_4 = \partial(P_4 - P_3)$$

$$P_1 = (0,0), P_2 = (1,0), P_3 = (2,0), P_4 = (3,0)$$

$$P_4 - P_1 = R_1 = R_4 = (3,0)$$
$$\partial = 3$$

Bezier Curves

$$G_B = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Bezier Curves

$$R_1 = 3(P_2 - P_1)$$

$$R_4 = 3(P_4 - P_3)$$

$$G_H = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \bullet \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = M_{HB} \bullet G_B$$

Bezier Curves

$$
\begin{aligned}
Q(t) &= T \bullet M_H \bullet G_H \\
&= T \bullet M_H \bullet (M_{HB} \bullet G_B) \\
&= T \bullet (M_H \bullet M_{HB}) \bullet G_B \\
&= T \bullet M_B \bullet G_B
\end{aligned}
$$

$$
\begin{aligned}
M_B &= M_H \bullet M_{HB} \\
&= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Bezier Curves

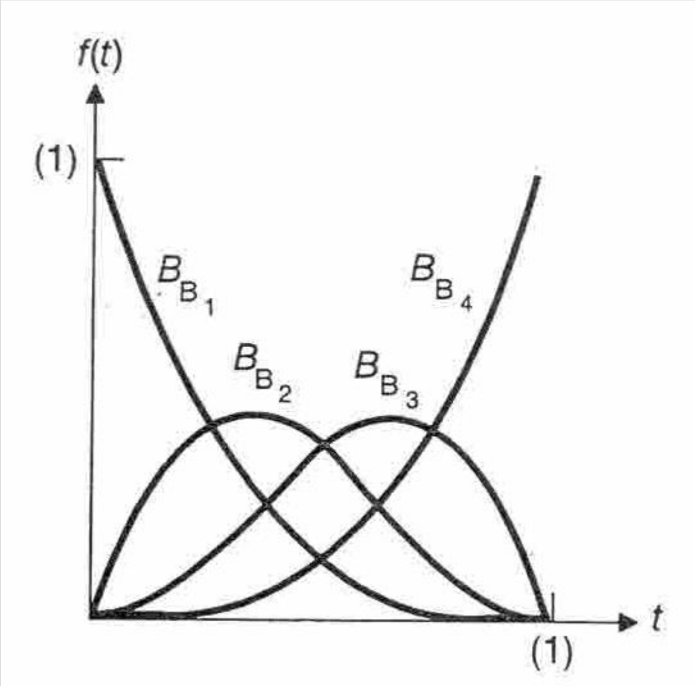
$$Q(t) = T \bullet M_B \bullet G_B = B_B \bullet G_B$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{2x} & P_{2y} & P_{2z} \\ P_{3x} & P_{3y} & P_{3z} \\ P_{4x} & P_{4y} & P_{4z} \end{bmatrix}$$

$$= (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

Bezier Curves

$$B_B = \begin{bmatrix} -t^3 + 3t^2 - 3t + 1 & 3t^3 - 6t^2 + 3t & -3t^3 + 3t^2 & t^3 \end{bmatrix}$$
$$= \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix}$$



Bezier Curves

- Define *n* as the order of Bezier curves.
- Define *i* as control point.

$$B_{i,n}(t) = C(n,i) \bullet t^i \bullet (1-t)^{n-i}$$

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

$$Q(t) = \sum_{i=0}^{n} B_{i,n}(t) P_i$$

$$= \sum_{i=0}^{n} C(n,i) \bullet u^i \bullet (1-t)^{n-i}$$

Bezier Curves

$$B_{0,3}(t) = \frac{3!}{0! \bullet 3!} \bullet t^0 \bullet (1-t)^{3-0} = (1-t)^3$$

$$B_{1,3}(t) = \frac{3!}{1! \bullet 2!} \bullet t^1 \bullet (1-t)^{3-1} = 3t(1-t)^2$$

$$B_{2,3}(t) = \frac{3!}{2! \bullet 1!} \bullet t^2 \bullet (1-t)^{3-2} = 3t^2(1-t)$$

$$B_{3,3}(t) = \frac{3!}{3! \bullet 0!} \bullet t^3 \bullet (1-t)^{3-3} = t^3$$

Bezier Curves

$$Q(t) = (1-t)^3 \bullet P_0 + 3t(1-t)^2 \bullet P_1 + 3t^2(1-t) \bullet P_2 + t^3 \bullet P_3$$

$$= \begin{bmatrix}(1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3\end{bmatrix} \bullet \begin{bmatrix}P_0 \\ P_1 \\ P_2 \\ P_3\end{bmatrix}$$

$$= \begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix} \bullet \begin{bmatrix}-1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0\end{bmatrix} \bullet \begin{bmatrix}P_0 \\ P_1 \\ P_2 \\ P_3\end{bmatrix}$$

$$= T \bullet M_B \bullet G_B$$

Bezier Curves

- Linear Bezier splines
  Control points: P_0, P_1

$$P(t) = P_0 + (P_1 - P_0)t = (1-t)P_0 + tP_1$$

$$0 \le t \le 1$$

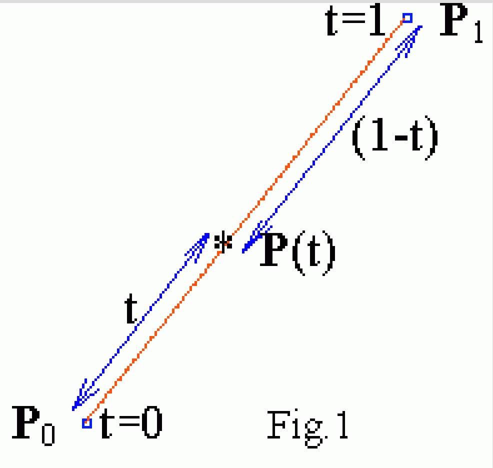


Fig. 1

Bezier Curves

- Quadratic Bezier splines
  Control points: P_0, P_1, P_2

$$P_0^1 = P_0 + (P_1 - P_0)t = (1-t)P_0 + tP_1$$

$$P_1^1 = P_1 + (P_2 - P_1)t = (1-t)P_1 + tP_2$$

$$0 \le t \le 1$$

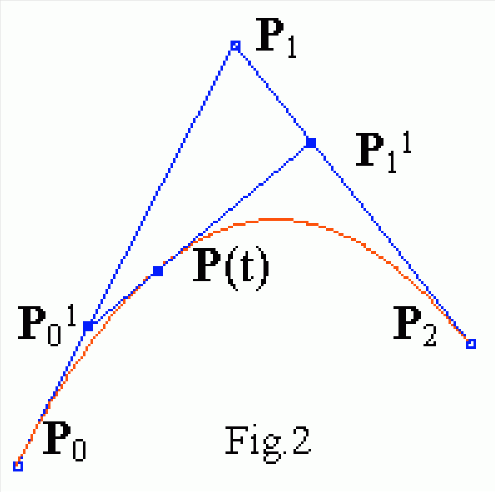


Fig.2

Bezier Curves

- Quadratic Bezier splines
  Control points: P_0, P_1, P_2

$$P(t) = P_0^{\,1} + (P_1^{\,1} - P_0^{\,1})t$$
$$= (1-t)P_0^{\,1} + tP_1^{\,1}$$
$$= (1-t)\big[(1-t)P_0 + tP_1\big] + t\big[(1-t)P_1 + tP_2\big]$$
$$= (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$$

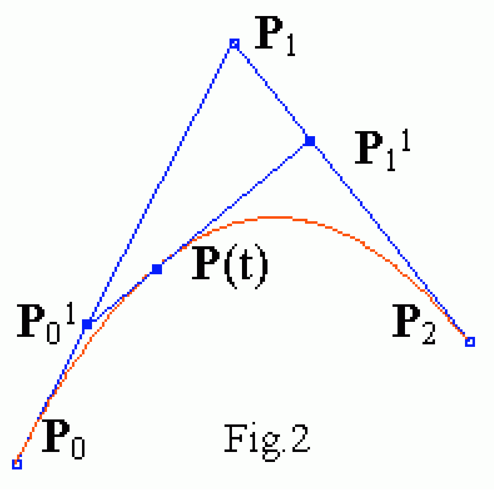


Fig.2

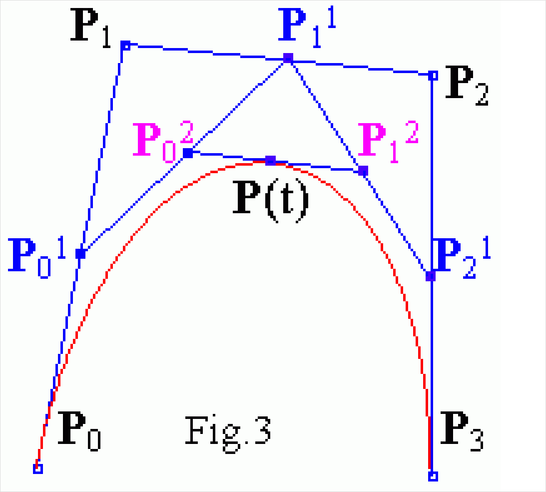
Bezier Curves

- Cubic Bezier splines
  Control points: P_0, P_1, P_2, P_3

$$P_0^1 = P_0 + (P_1 - P_0)t = (1-t)P_0 + tP_1$$

$$P_1^1 = P_1 + (P_2 - P_1)t = (1-t)P_1 + tP_2$$

$$P_2^1 = P_2 + (P_3 - P_2)t = (1-t)P_2 + tP_3$$



Bezier Curves

- Cubic Bezier splines
  Control points: P_0, P_1, P_2, P_3

$$
\begin{aligned}
P_0^{\,2} &= P_0^{\,1} + (P_1^{\,1} - P_0^{\,1})t = (1-t)P_0^{\,1} + tP_1^{\,1} \\
&= (1-t)\big[(1-t)P_0 + tP_1\big] + t\big[(1-t)P_1 + tP_2\big] \\
&= (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2
\end{aligned}
$$

$$
\begin{aligned}
P_1^{\,2} &= P_1^{\,1} + (P_2^{\,1} - P_1^{\,1})t = (1-t)P_1^{\,1} + tP_2^{\,1} \\
&= (1-t)\big[(1-t)P_1 + tP_2\big] + t\big[(1-t)P_2 + tP_3\big] \\
&= (1-t)^2 P_1 + 2t(1-t)P_2 + t^2 P_3
\end{aligned}
$$

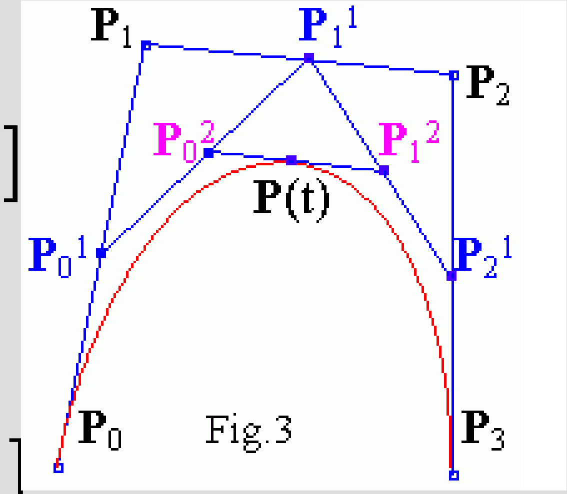


Fig.3

Bezier Curves

- Cubic Bezier splines
  Control points: P_0, P_1, P_2, P_3

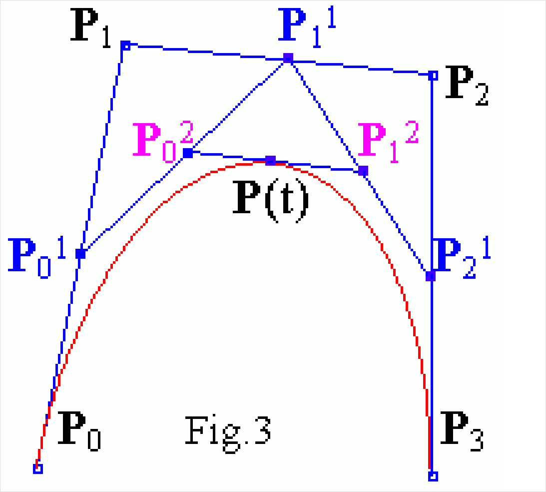


Fig.3

$$
\begin{aligned}
P(t) &= P_0^{\ 2} + (P_1^{\ 2} - P_0^{\ 2})t \\
&= (1-t)P_0^{\ 2} + tP_1^{\ 2} \\
&= (1-t)\left[(1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2\right] + \\
&\qquad t\left[(1-t)^2 P_1 + 2t(1-t)P_2 + t^2 P_3\right] \\
&= (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3
\end{aligned}
$$

Bezier Curves

$$P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$$

$$= \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix} \bullet \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \bullet \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} P_{0x} & P_{0y} & P_{0z} \\ P_{1x} & P_{1y} & P_{1z} \\ P_{2x} & P_{2y} & P_{2z} \\ P_{3x} & P_{3y} & P_{3z} \end{bmatrix}$$

$$= T \bullet M_B \bullet G_B$$

Bezier Curves

- http://www.ibiblio.org/e-notes/Splines/Bezier.htm

Spline

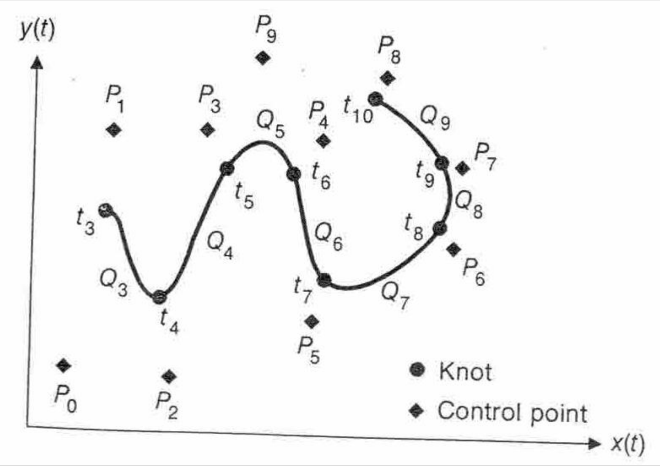
- Natural cubic spline
  - C^0, C^1, C^2 continuous.
  - Interpolates(passes through) the control points.
  - Moving any one control point affects the entire curve.

Spline

- B-spline
  - Local control.
  - Moving a control point affects only a small part of a curve.
  - Do not interpolate their control points.
  - Sharing control points between segments.

B-spline

- m+1 control points $P_0, \ldots, P_m$, $m \geq 3$
- m-2 curve segments $Q_3, Q_4, \ldots, Q_m$
- For each $i \geq 4$, there is a join point or knot between Q_{i-1} and Q_i at the parameter value t_i.



B-spline

$$B_{i,1}(t) = 1, t_i \le t < t_{i+1}$$

$$B_{i,1}(t) = 0, otherwise$$

$$B_{i,d}(t) = \frac{t - t_i}{t_{i+d-1} - t_i} B_{i,d-1}(t) + \frac{t_{i+d} - t}{t_{i+d} - t_{i+1}} B_{i+1,d-1}(t)$$

$$Q_i(t) = P_{i-3} \bullet B_{i-3,4}(t) + P_{i-2} \bullet B_{i-2,4}(t) + P_{i-1} \bullet B_{i-1,4}(t) + P_i \bullet B_{i,4}(t)$$

B-spline

$$B_{i,2}(t)=\frac{t-t_i}{t_{i+1}-t_i}B_{i,1}(t)+\frac{t_{i+2}-t}{t_{i+2}-t_{i+1}}B_{i+1,1}(t)$$

$$B_{i,3}(t)=\frac{t-t_i}{t_{i+2}-t_i}B_{i,2}(t)+\frac{t_{i+3}-t}{t_{i+3}-t_{i+1}}B_{i+1,2}(t)$$

$$B_{i,4}(t)=\frac{t-t_i}{t_{i+3}-t_i}B_{i,3}(t)+\frac{t_{i+4}-t}{t_{i+4}-t_{i+1}}B_{i+1,3}(t)$$

Uniform Nonrational B-spline

- '*Uniform*' means that the knots are spaced at equal intervals of the parameter t.
- '*Nonrational*' is used to distinguish these splines from rational cubic polynomial curves, see Section 11.2.5
- We assume that $t_3=0$ and the interval $t_{i+1}-t_i=1$

Uniform Nonrational B-spline

$$Q_i(t) = T_i \bullet M_{BS} \bullet G_{B_{Si}}, t_i \le t < t_{i+1}$$

$$G_{B_{Si}} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}, 3 \le i \le m$$

$$M_{Bs} = \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

Uniform Nonrational B-spline

$$
\begin{aligned}
B_{Bs} = T \bullet M_{Bs} &= \begin{bmatrix} B_{Bs-3} & B_{Bs-2} & B_{Bs-1} & B_{Bs_0} \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} -t^3+3t^2-3t+1 & 3t^3-6t^2+4 & -3t^3+3t^2+3t+1 & t^3 \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} (1-t)^3 & 3t^3-6t^2+4 & -3t^3+3t^2+3t+1 & t^3 \end{bmatrix}, 0 \le t < 1.
\end{aligned}
$$

Uniform Nonrational B-spline

$$Q_i(t) = T_i \bullet M_{Bs} \bullet G_{Bs_i}$$
$$= B_{Bs} \bullet G_{Bs_i} = B_{B_{S-3}} \bullet P_{i-3} + B_{B_{S-2}} \bullet P_{i-2} + B_{B_{S-1}} \bullet P_{i-1} + B_{B_{S0}} \bullet P_i$$
$$= \frac{(1-t)^3}{6} P_{i-3} + \frac{3t^3 - 6t^2 + 4}{6} P_{i-2} + \frac{-3t^3 + 3t^2 + 3t + 1}{6} P_{i-1} + \frac{t^3}{6} P_i$$

$$t_i \le t < t_{i+1}$$

Uniform Nonrational B-spline

$$x_i(t_{i+1}) = x_{i+1}(t_{i+1})$$

$$\frac{d}{dt} x_i(t_{i+1}) = \frac{d}{dt} x_{i+1}(t_{i+1})$$

$$\frac{d^2}{dt^2} x_i(t_{i+1}) = \frac{d^2}{dt^2} x_{i+1}(t_{i+1})$$

Uniform Nonrational B-spline

$$x_i \,\big|_{t-t_i=1} = x_{i+1} \,\big|_{t-t_{i+1}=0}$$

$$\frac{d}{dt} x_i \,\big|_{t-t_i=1} = \frac{d}{dt} x_{i+1} \,\big|_{t-t_{i+1}=0}$$

$$\frac{d^2}{dt^2} x_i \,\big|_{t-t_i=1} = \frac{d^2}{dt^2} x_{i+1} \,\big|_{t-t_{i+1}=0}$$

Uniform Nonrational B-spline

$$Q_i(t-t_i)=\frac{1-3t+3t^2-t^3}{6}P_{i-3}+\frac{3t^3-6t^2+4}{6}P_{i-2}+\frac{-3t^3+3t^2+3t+1}{6}P_{i-1}+\frac{t^3}{6}P_i$$

$$Q_i'(t-t_i)=\frac{-3+6t-3t^2}{6}P_{i-3}+\frac{9t^2-12t}{6}P_{i-2}+\frac{-9t^2+6t+3}{6}P_{i-1}+\frac{3t^2}{6}P_i$$

$$Q_i''(t-t_i)=\frac{6-6t}{6}P_{i-3}+\frac{18t-12}{6}P_{i-2}+\frac{-18t+6}{6}P_{i-1}+\frac{6t}{6}P_i$$

$$=(1-t)P_{i-3}+(3t-2)P_{i-2}+(-3t+1)P_{i-1}+tP_i$$

Uniform Nonrational B-spline

$$Q_{i+1}(t-t_{i+1}) = \frac{1-3t+3t^2-t^3}{6}P_{i-2} + \frac{3t^3-6t^2+4}{6}P_{i-1} + \frac{-3t^3+3t^2+3t+1}{6}P_i + \frac{t^3}{6}P_{i+1}$$

$$Q_{i+1}'(t-t_{i+1}) = \frac{-3+6t-3t^2}{6}P_{i-2} + \frac{9t^2-12t}{6}P_{i-1} + \frac{-9t^2+6t+3}{6}P_i + \frac{3t^2}{6}P_{i+1}$$

$$Q_{i+1}''(t-t_{i+1}) = \frac{6-6t}{6}P_{i-2} + \frac{18t-12}{6}P_{i-1} + \frac{-18t+6}{6}P_i + \frac{6t}{6}P_{i+1}$$

$$= (1-t)P_{i-2} + (3t-2)P_{i-1} + (-3t+1)P_i + tP_{i+1}$$

Uniform Nonrational B-spline

$$x_i \,|_{t-t_i=1} = x_{i+1} \,|_{t-t_{i+1}=0} = Q_i(1) = Q_{i+1}(0) = \frac{P_{i-2_x} + 4P_{i-1_x} + P_{i_x}}{6}$$

$$\frac{d}{dt} x_i \,|_{t-t_i=1} = \frac{d}{dt} x_{i+1} \,|_{t-t_{i+1}=0} = Q_i(1) = Q_{i+1}(0) = \frac{-P_{i-2_x} + P_{i_x}}{2}$$

$$\frac{d^2}{dt^2} x_i \,|_{t-t_i=1} = \frac{d^2}{dt^2} x_{i+1} \,|_{t-t_{i+1}=0} = Q_i(1) = Q_{i+1}(0) = P_{i-2_x} - 2P_{i-1_x} + P_{i_x}$$

Uniform Nonrational B-spline

- The curve can be forced to be interpolate specific points by replicating control points.

$$P_{i-2} = P_{i-1} = P_i,$$

$$Q_i(t) = B_{Bs-3} \bullet P_{i-3} + (B_{Bs-2} + B_{Bs-1} + B_{Bs0}) \bullet P_i$$

Nonuniform Nonrational B-spline

- Parameter interval between successive knot values need not be uniform.
- Blending functions are no longer the same for each interval.
- Continuity at selected join points can be reduced from C^2 to C^1 to C^0 to none.
  - When the curve is C^0, the curve interpolates a control point.

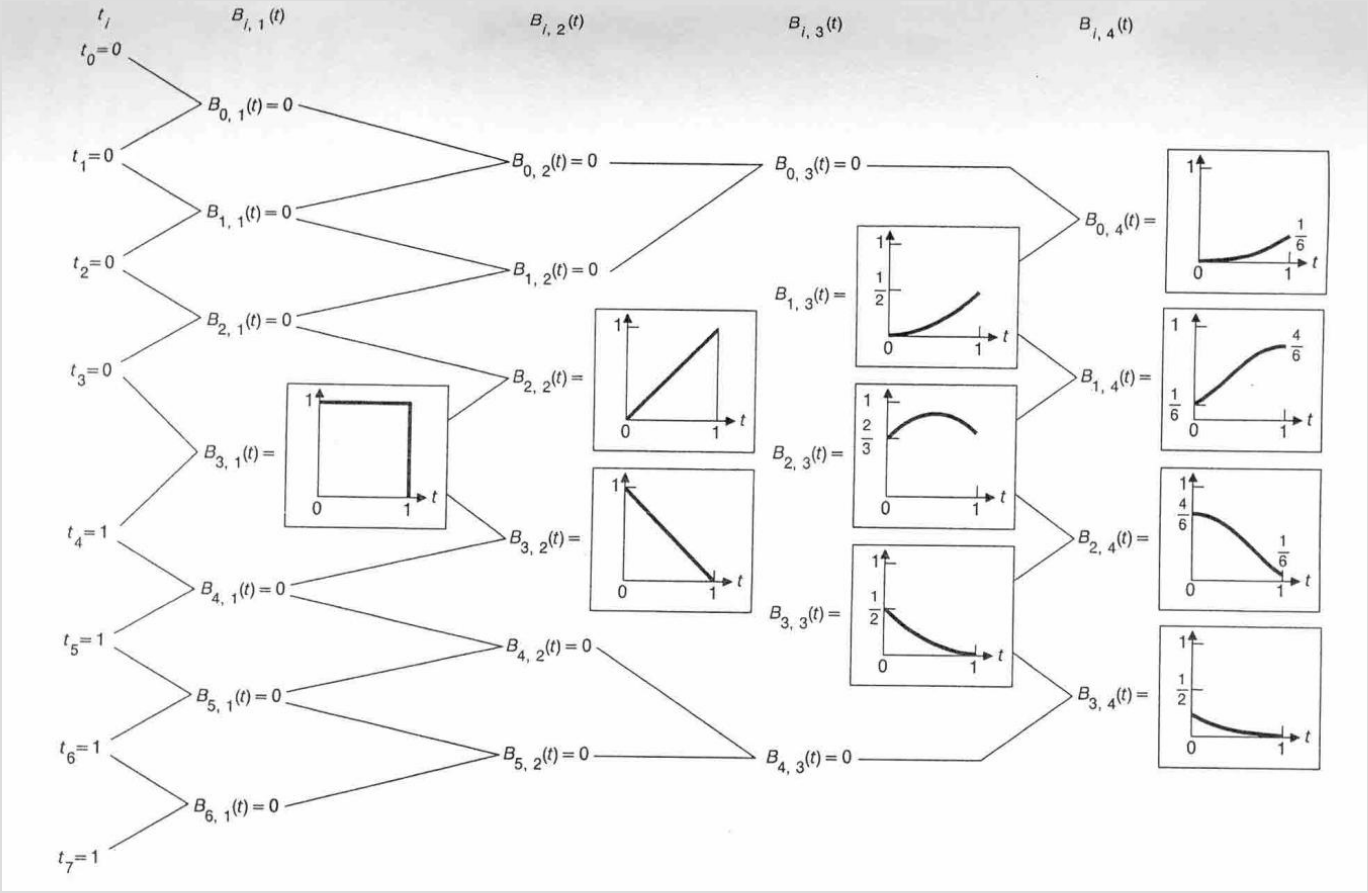
Nonuniform Nonrational B-spline

- If the continuity is reduced to C^0, then the curve interpolates a control point, but without the undesirable effect of uniform B-splines, where the curve segments on either side of the interpolated control point are straight lines.

Nonuniform Nonrational B-spline

- m+1 control points $P_0, P_1, \ldots, P_m$
- nondecreasing sequence of knot values $t_0, t_1, \ldots, t_{m+4}$

Nonuniform Nonrational B-spline



Nonuniform Rational Cubic Polynomial Curve

$$x(t) = \frac{X(t)}{W(t)}, y(t) = \frac{Y(t)}{W(t)}, z(t) = \frac{Z(t)}{W(t)}$$

$$Q(t) = \begin{bmatrix} X(t) & Y(t) & Z(t) & W(t) \end{bmatrix}$$

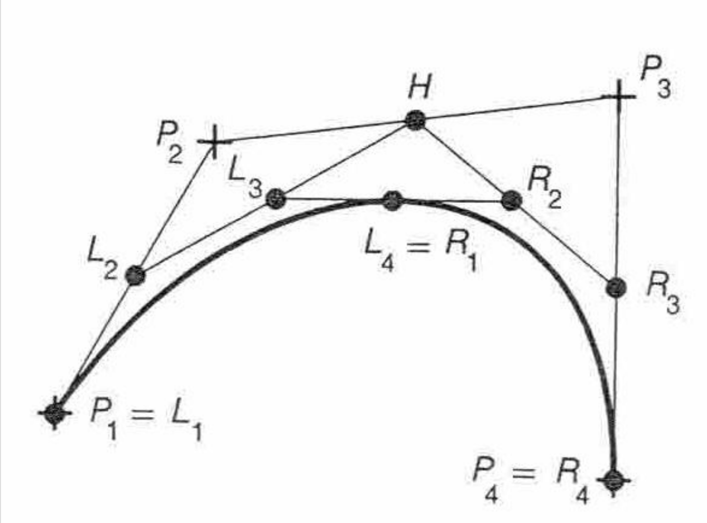
Nonuniform Rational Cubic Polynomial Curve

- Advantages
  - They are invariant under rotation, scaling, translation and perspective transformations of the control points.
  - They can define precisely and of the conic sections.

Subdividing Curves

- Bezier Curve

$$L_2 = (P_1 + P_2)/2$$
$$H = (P_2 + P_3)/2$$
$$L_3 = (L_2 + H)/2$$
$$R_3 = (P_3 + P_4)/2$$
$$R_2 = (H + R_3)/2$$
$$L_4 = R_1 = (L_3 + R_2)/2$$



Subdividing Curves

$$G_B^L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} = D_B^L \bullet G_B = \frac{1}{8}\begin{bmatrix} 8 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \bullet \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

$$G_B^R = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = D_B^R \bullet G_B = \frac{1}{8}\begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{bmatrix} \bullet \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Subdividing Curves

- Uniform B-spline Curve
  - Doubling: $P_0, P_0, P_1, P_1, P_2, P_2, P_3, P_3$
  - Do *k* subdivision steps (for *k*-degree B-spline)

$$P_0, \frac{P_0+P_1}{2}, P_1, \frac{P_1+P_2}{2}, P_2, \frac{P_2+P_3}{2}, P_3$$

$$\frac{3P_0+P_1}{4}, \frac{P_0+3P_1}{4}, \frac{3P_1+P_2}{4}, \frac{P_1+3P_2}{4}, \frac{3P_2+P_3}{4}, \frac{P_2+3P_3}{4}$$

$$\frac{P_0+P_1}{2}, \frac{P_0+6P_1+P_2}{8}, \frac{P_1+P_2}{2}, \frac{P_1+6P_2+P_3}{8}, \frac{P_2+P_3}{2}$$

Subdividing Curves

$$G_{B\,Si}^{L} = D_{B\,S}^{L} \bullet G_{B\,Si} = \frac{1}{8}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}\begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_{i} \end{bmatrix}$$

$$G_{B\,Si}^{R} = D_{B\,S}^{R} \bullet G_{BSi} = \frac{1}{8}\begin{bmatrix} 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 4 & 4 \end{bmatrix}\begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_{i} \end{bmatrix}$$

Conversion

$$T \bullet M_2 \bullet G_2 = T \bullet M_1 \bullet G_1$$

$$M_2 \bullet G_2 = M_1 \bullet G_1$$

$$M_2^{-1} \bullet M_2 \bullet G_2 = G_2 = M_2^{-1} \bullet M_1 \bullet G_1$$

Conversion

$$T \bullet M_B \bullet G_B = T \bullet M_{Bs} \bullet G_{Bs}$$

$$G_B = M_B^{-1} \bullet M_{Bs} \bullet G_{Bs} = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \bullet G_{Bs}$$

$$G_{Bs} = M_{Bs}^{-1} \bullet M_B \bullet G_B = \begin{bmatrix} 6 & -7 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -7 & 6 \end{bmatrix} \bullet G_B$$

Comparison

TABLE 11.2 COMPARISON OF SEVEN DIFFERENT FORMS OF PARAMETRIC CUBIC CURVES

	Hermite	Bézier	Uniform B-spline	Uniformly shaped β-spline	Nonuniform B-spline	Catmull–Rom	Kochanek–Bartels
Convex hull defined by control points	N/A	Yes	Yes	Yes	Yes	No	No
Interpolates some control points	Yes	Yes	No	No	No	Yes	Yes
Interpolates all control points	Yes	No	No	No	No	Yes	Yes
Ease of subdivision	Good	Best	Avg	Avg	High	Avg	Avg
Continuities inherent in representation	C^0 G^0	C^0 G^0	C^2 G^2	C^0 G^2	C^2 G^2	C^1 G^1	C^1 G^1
Continuities easily achieved	C^1 G^1	C^1 G^1	C^2 G^{2*}	C^1 G^{2*}	C^2 G^{2*}	C^1 G^1	C^1 G^1
Number of parameters controlling a curve segment	4	4	4	6†	5	4	7

*Except for special case discussed in Section 11.2.
†Four of the parameters are local to each segment, two are global to the entire curve.

Parametric Bicubic Surfaces

$$Q(t) = T \bullet M \bullet G$$

$$Q(s) = S \bullet M \bullet G$$

$$Q(s,t) = S \bullet M \bullet G(t) = S \bullet M \bullet \begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \end{bmatrix}$$

Parametric Bicubic Surfaces

$$G_i(t) = T \bullet M \bullet G_i = T \bullet M \bullet \begin{bmatrix} g_{i1} \\ g_{i2} \\ g_{i3} \\ g_{i4} \end{bmatrix}$$

$$= T \bullet M \bullet \begin{bmatrix} g_{i1} & g_{i2} & g_{i3} & g_{i4} \end{bmatrix}^T$$

$$G_i(t)^T = \begin{bmatrix} g_{i1} & g_{i2} & g_{i3} & g_{i4} \end{bmatrix} \bullet M^T \bullet T^T$$

Parametric Bicubic Surfaces

$$Q(s,t) = S \bullet M \bullet \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \bullet M^T \bullet T^T$$

$$= S \bullet M \bullet G \bullet M^T \bullet T^T, 0 \le s,t \le 1.$$

Parametric Bicubic Surfaces

$$x(s,t) = S \bullet M \bullet G_x \bullet M^T \bullet T^T$$

$$y(s,t) = S \bullet M \bullet G_y \bullet M^T \bullet T^T$$

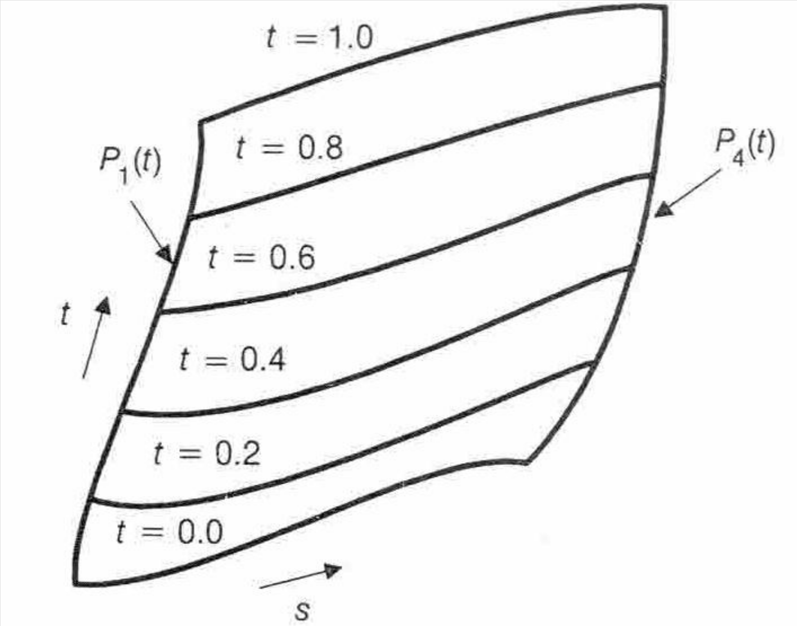
$$z(s,t) = S \bullet M \bullet G_z \bullet M^T \bullet T^T$$

Hermite Surfaces

$$x(s,t) = S \bullet M_H \bullet G_{H_x}(t)$$

$$= S \bullet M_H \bullet \begin{bmatrix} P_1(t) \\ P_4(t) \\ R_1(t) \\ R_4(t) \end{bmatrix}_x$$

Hermite Surfaces



Hermite Surfaces

$$P_{1x}(t) = T \bullet M_H \bullet \begin{bmatrix} g_{11} \\ g_{12} \\ g_{13} \\ g_{14} \end{bmatrix}_x, P_{4x}(t) = T \bullet M_H \bullet \begin{bmatrix} g_{21} \\ g_{22} \\ g_{23} \\ g_{24} \end{bmatrix}_x$$

$$R_{1x}(t) = T \bullet M_H \bullet \begin{bmatrix} g_{31} \\ g_{32} \\ g_{33} \\ g_{34} \end{bmatrix}_x, R_{4x}(t) = T \bullet M_H \bullet \begin{bmatrix} g_{41} \\ g_{42} \\ g_{43} \\ g_{44} \end{bmatrix}_x$$

Hermite Surfaces

$$\begin{bmatrix} P_1(t) & P_4(t) & R_1(t) & R_4(t) \end{bmatrix}_x = T \bullet M_H \bullet G_{H_x}^T$$

$$G_{H_x} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}_x$$

$$\begin{bmatrix} P_1(t) \\ P_4(t) \\ R_1(t) \\ R_4(t) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}_x \bullet M_H^T \bullet T^T = G_{H_x} \bullet M_H^T \bullet T^T$$

Hermite Surfaces

$$x(s,t) = S \bullet M_H \bullet G_{H_x} \bullet M_H^T \bullet T^T$$

$$y(s,t) = S \bullet M_H \bullet G_{H_y} \bullet M_H^T \bullet T^T$$

$$z(s,t) = S \bullet M_H \bullet G_{H_z} \bullet M_H^T \bullet T^T$$

$$Q(s,t) = S \bullet M_H \bullet G_H \bullet M_H^T \bullet T^T$$

Hermite Surfaces

$$G_{H_x} = \begin{bmatrix} x(0,0) & x(0,1) & \frac{\partial}{\partial t} x(0,0) & \frac{\partial}{\partial t} x(0,1) \\ x(1,0) & x(1,1) & \frac{\partial}{\partial t} x(1,0) & \frac{\partial}{\partial t} x(1,1) \\ \frac{\partial}{\partial s} x(0,0) & \frac{\partial}{\partial s} x(0,1) & \frac{\partial^2}{\partial s \partial t} x(0,0) & \frac{\partial^2}{\partial s \partial t} x(0,1) \\ \frac{\partial}{\partial s} x(1,0) & \frac{\partial}{\partial s} x(1,1) & \frac{\partial^2}{\partial s \partial t} x(1,0) & \frac{\partial^2}{\partial s \partial t} x(1,1) \end{bmatrix}$$

Normals to Surfaces

$$\frac{\partial}{\partial s}Q(s,t) = \frac{\partial}{\partial s}(S \bullet M \bullet G \bullet M^T \bullet T^T)$$

$$= \frac{\partial}{\partial s}(S) \bullet M \bullet G \bullet M^T \bullet T^T$$

$$= \begin{bmatrix} 3s^2 & 2s & 1 & 0 \end{bmatrix} \bullet M \bullet G \bullet M^T \bullet T^T$$

$$\frac{\partial}{\partial t}Q(s,t) = \frac{\partial}{\partial t}(S \bullet M \bullet G \bullet M^T \bullet T^T)$$

$$= S \bullet M \bullet G \bullet M^T \bullet \frac{\partial}{\partial t}(T^T)$$

$$= S \bullet M \bullet G \bullet M^T \bullet \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix}^T$$

Normals to Surfaces

$$\frac{\partial}{\partial s}Q(s,t)=(x_s,y_s,z_s)$$

$$\frac{\partial}{\partial t}Q(s,t)=(x_t,y_t,z_t)$$

$$\frac{\partial}{\partial s}Q(s,t)\times\frac{\partial}{\partial t}Q(s,t)=\begin{bmatrix}y_s z_t - y_t z_s & z_s x_t - z_t x_s & x_s y_t - x_t y_s\end{bmatrix}$$

Thank You!