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Discrete Mathematics and Its Applications

Sixth Edition

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Chapter 2
Basic Structures: Sets, Functions, Sequences, and Sums

Outlines

1. Sets
2. Set Operations
3. Functions
4. Sequences and Summations

Getting Started

1. Sets
2. Set Operations
3. Functions
4. Sequences and Summations

2.1 Sets(1/8)

- **Definition 1:** A ***set*** is an unordered collection of objects
- **Definition 2:** Objects in a set are called ***elements***, or ***members*** of the set.
  - $a \in A$, $a \notin A$
  - $V = \{a, e, i, o, u\}$
  - $O = \{1, 3, 5, 7, 9\}$
    or $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$
    or $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

2.1 Sets(2/8)

- $\mathbf{N}=\{0, 1, 2, 3, \ldots\}$, natural numbers
- $\mathbf{Z}=\{\ldots,-2, -1, 0, 1, 2, \ldots\}$, integers
- $\mathbf{Z}^+=\{1, 2, 3, \ldots\}$, positive integers
- $\mathbf{Q}=\{p/q \mid p\in\mathbf{Z}, q\in\mathbf{Z}, \text{ and } q\neq 0\}$, rational numbers
- $\mathbf{Q}^+=\{x\in \mathrm{R} \mid x=p/q, \text{ for positive integers } p \text{ and } q\}$
- $\mathbf{R}$, real numbers

2.1 Sets(3/8)

- **Definition 3:** Two sets are ***equal*** **if and only if** they have the same elements.
  $A = B$ **iff** $\forall x (x \in A \leftrightarrow x \in B)$
- Venn diagram
  - Universal set U
  - Empty set (null set) $\varnothing$ (or {})

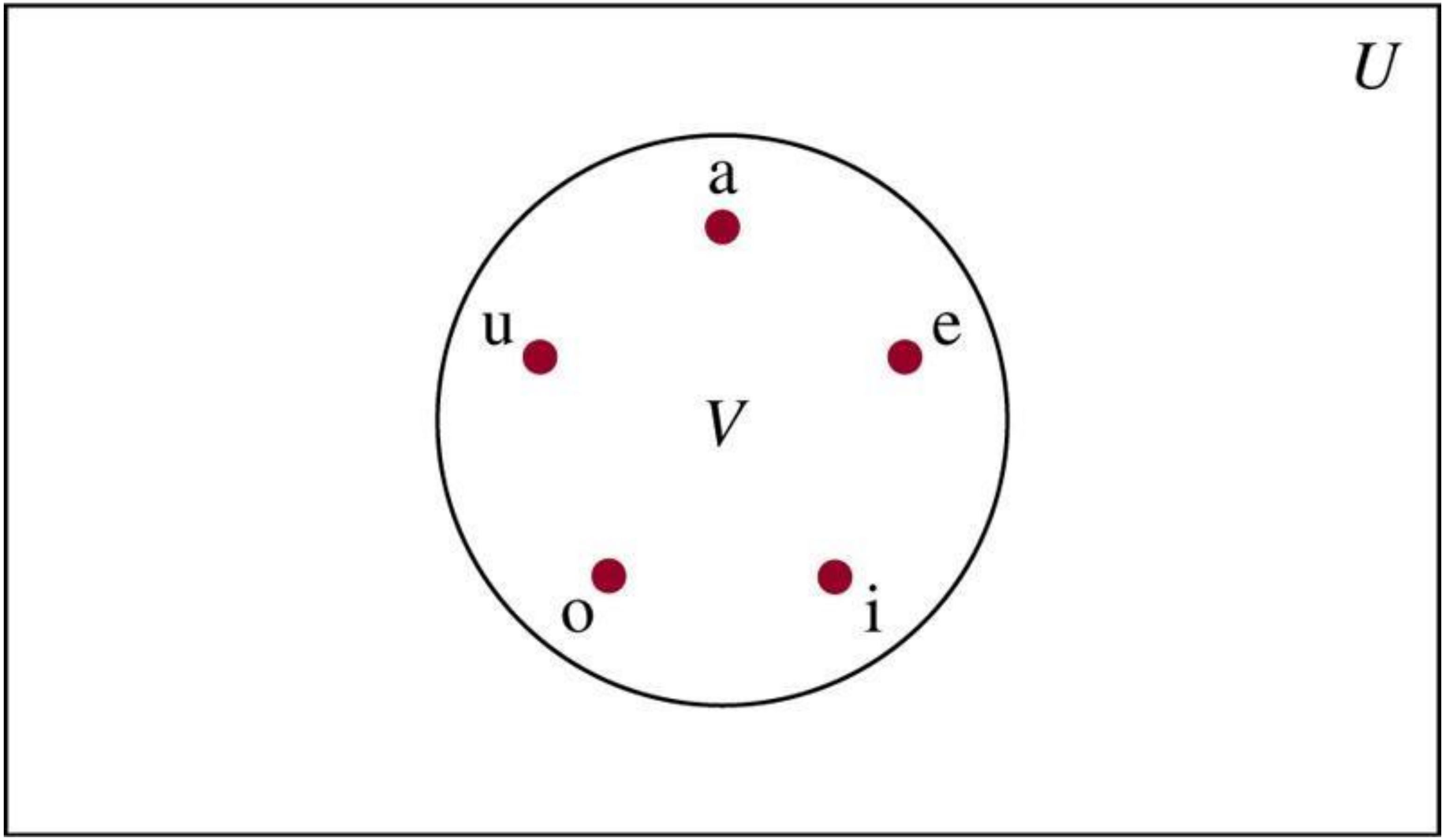

FIGURE 1 Venn Diagram for the Set of Vowels.

2.1 Sets(5/8)

- **Definition 4:** The set A is a ***subset*** of B **if and only if** every element of A is also an element of B.
  $A \subseteq B$ iff $\forall x(x \in A \rightarrow x \in B)$
- Theorem 1: For every set S,
  (1) $\varnothing \subseteq S$ and (2) $S \subseteq S$.
- Proper subset: $A \subset B$
  $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$

2.1 Sets(6/8)

- If $A \subseteq B$ and $B \subseteq A$, then $A=B$
- Sets may have other sets as members
  - $A=\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$
    $B=\{x|x$ is a subset of the set $\{a,b\}\}$
  - $A=B$

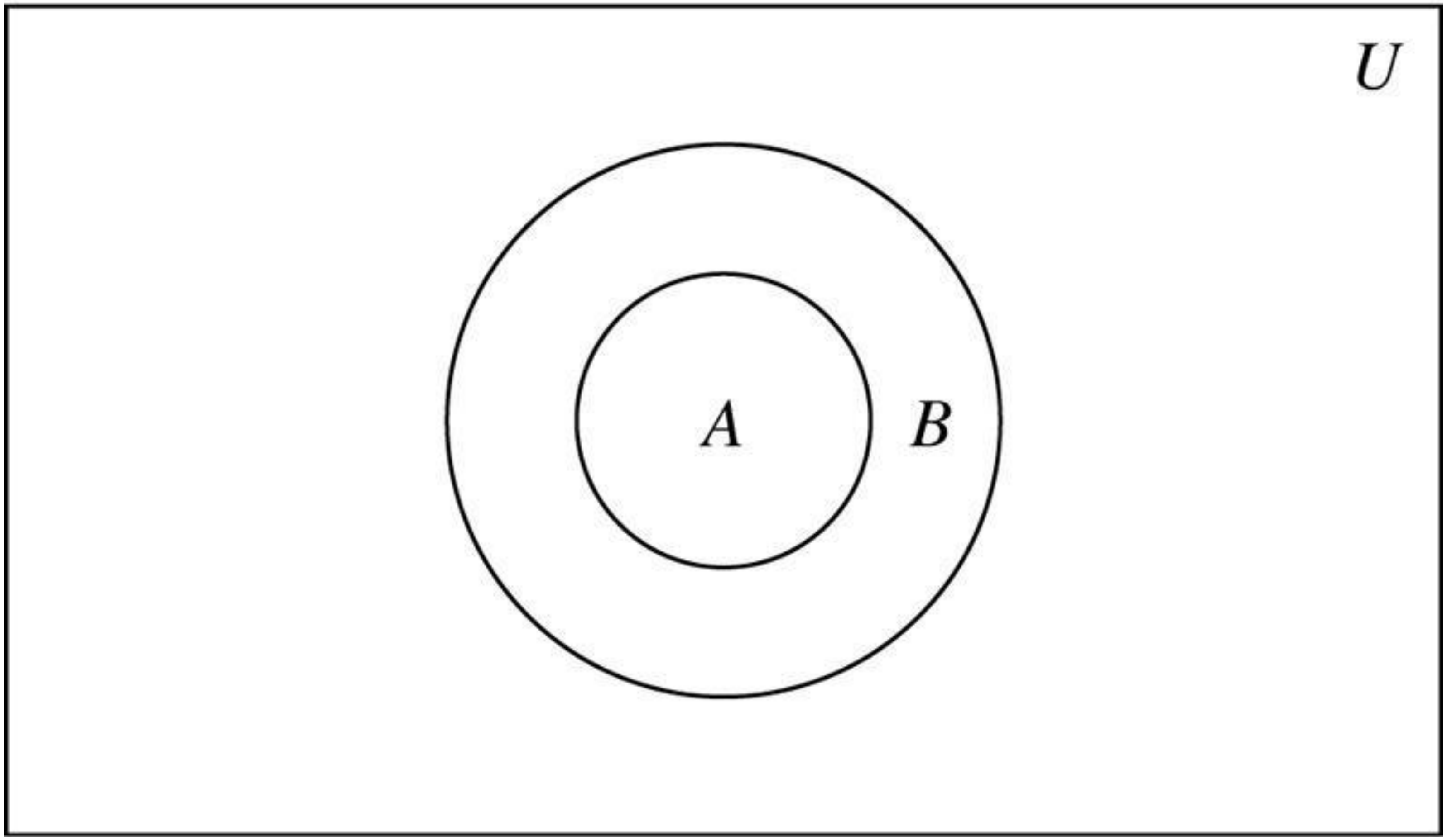

FIGURE 2 Venn Diagram Showing that A Is a Subset of B.

2.1 Sets(8/8)

- **Definition 5:** If there are exactly n distinct members in the set S (n is a nonnegative integer), we say that S is a finite set and that n is the ***cardinality*** of S.
  $|S| = n$
  - $|\varnothing| = 0$
- **Definition 6:** A set is ***infinite*** if it's not finite.
  - $\mathbf{Z}^+$

The Power Set

- **Definition 7:** The ***power set*** of S is the set of all subset of the set S. $P(S)$
  - $P(\{0,1,2\})$
  - $P(\emptyset)$
  - $P(\{\emptyset\})$
- If a set has n elements, then its subset has 2^n elements.

Cartesian Products

- **Definition 8:** ***Ordered n-tuple*** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_i as its ith element for $i=1, 2, \ldots, n$.
- **Definition 9:** ***Cartesian product*** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.
  $A \times B = \{(a, b) | a \in A \wedge b \in B\}$
  - *E.g.* $A = \{1, 2\}$, $B = \{a, b, c\}$
  - $A \times B$ and $B \times A$ are not equal, unless A=∅ or B=∅ or A=B

- **Definition 10:** *Cartesian product* of A_1, A_2, …, A_n, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i=1,2,\ldots,n$.
  $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i$ for $i=1,2,\ldots,n\}$

- Using set notation with quantifiers
  - $\forall x \in S\ (P(x))$: $\forall x\ (x \in S \rightarrow P(x))$
  - $\exists x \in S\ (P(x))$: $\exists x\ (x \in S \wedge P(x))$
  - $\exists x : x^2 = 4$ is true, since 2 is an x for which $x^2 = 4$. On the other hand, $\forall x : x^2 = 4$ is clearly false; not all numbers, when squared, are equal to 4.

- Truth sets of quantifiers

Given a predicate P, and a domain D, we define the truth set of P to be the set of elements x in D for which P (x) is true.

Then the truth set of *P*: $\{x \in D \mid P(x)\}$

- For Example:

What are the truth sets of the predicates P (x), where the domain is the set of integers and P (x) is "|x| = 1".

Solution: The truth set of P, $\{x \in Z \mid |x| = 1\}$, is the set of integers for which |x| = 1. Because |x| = 1 when x = 1 or x = −1, and for no other integers x, we see that the truth set of P is the set {−1, 1}.

Getting Started

1. Sets
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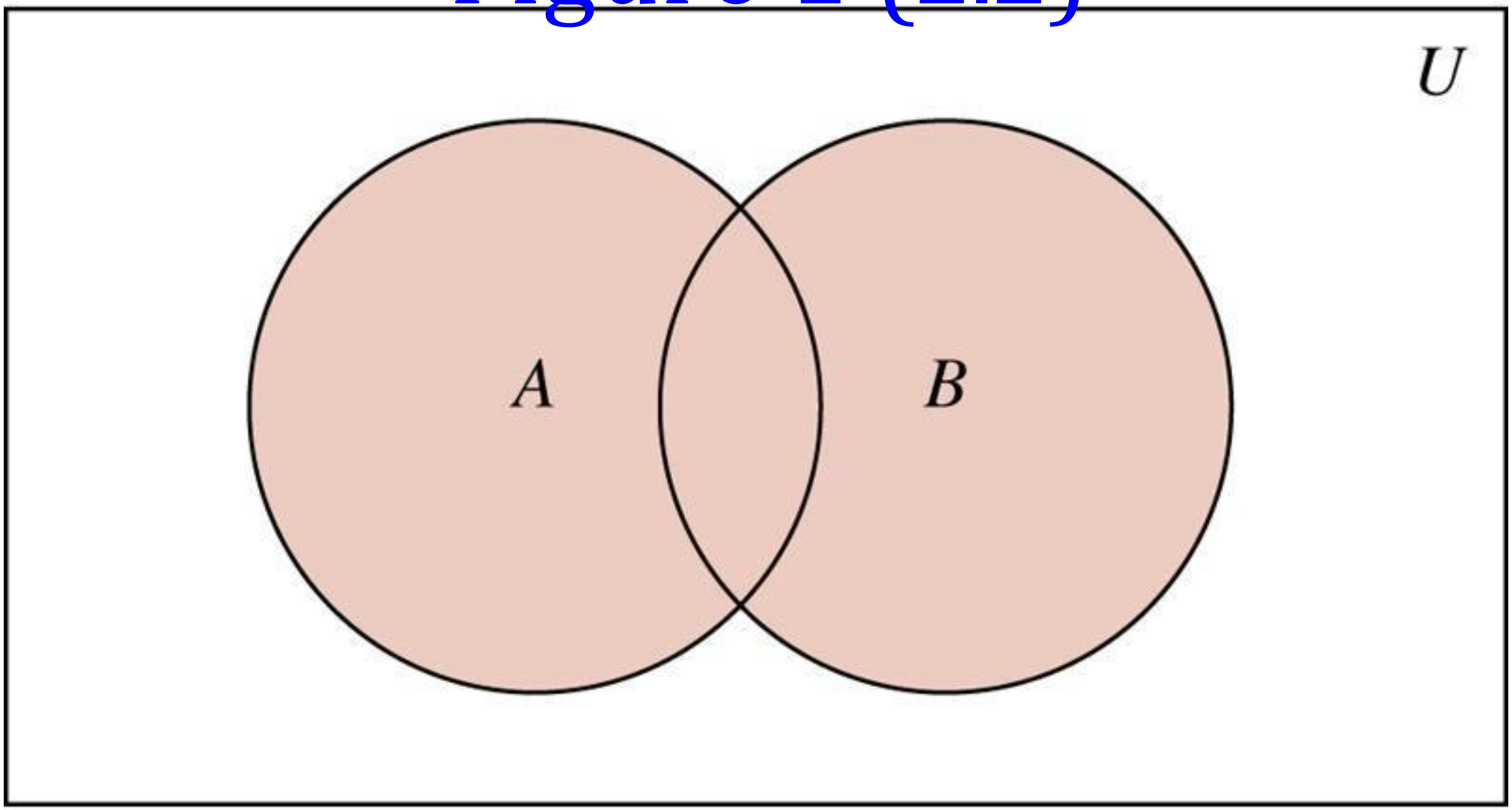
2.2 Set Operations

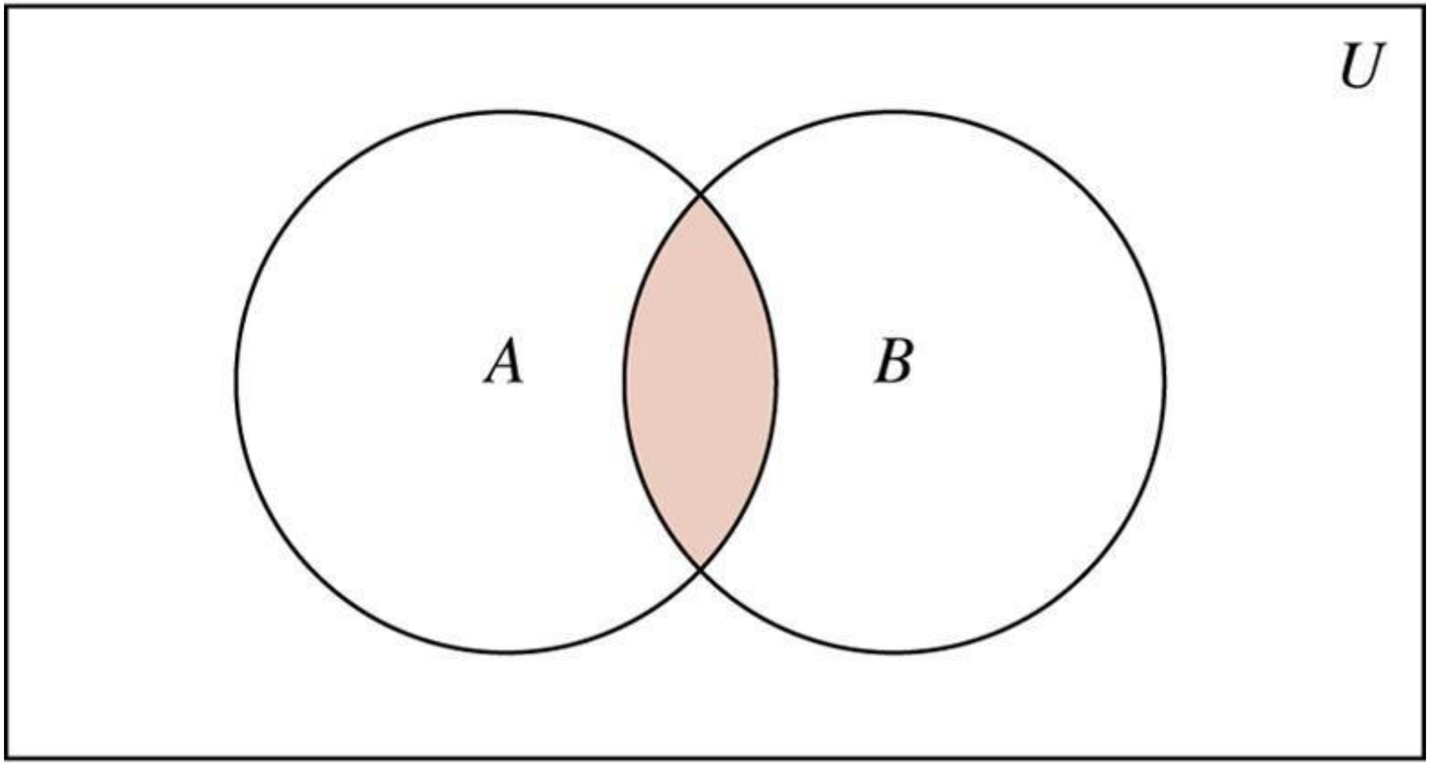
- **Definition 1:** The ***union*** of the sets A and B, denoted by $A \cup B$, is the set containing those elements that are either in A or in B, or in both.
  - $A \cup B = \{x | x \in A \vee x \in B\}$
- **Definition 2:** The ***intersection*** of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
  - $A \cap B = \{x | x \in A \wedge x \in B\}$

Figure 1 (2.2)



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram Representing the Union of *A* and *B*.

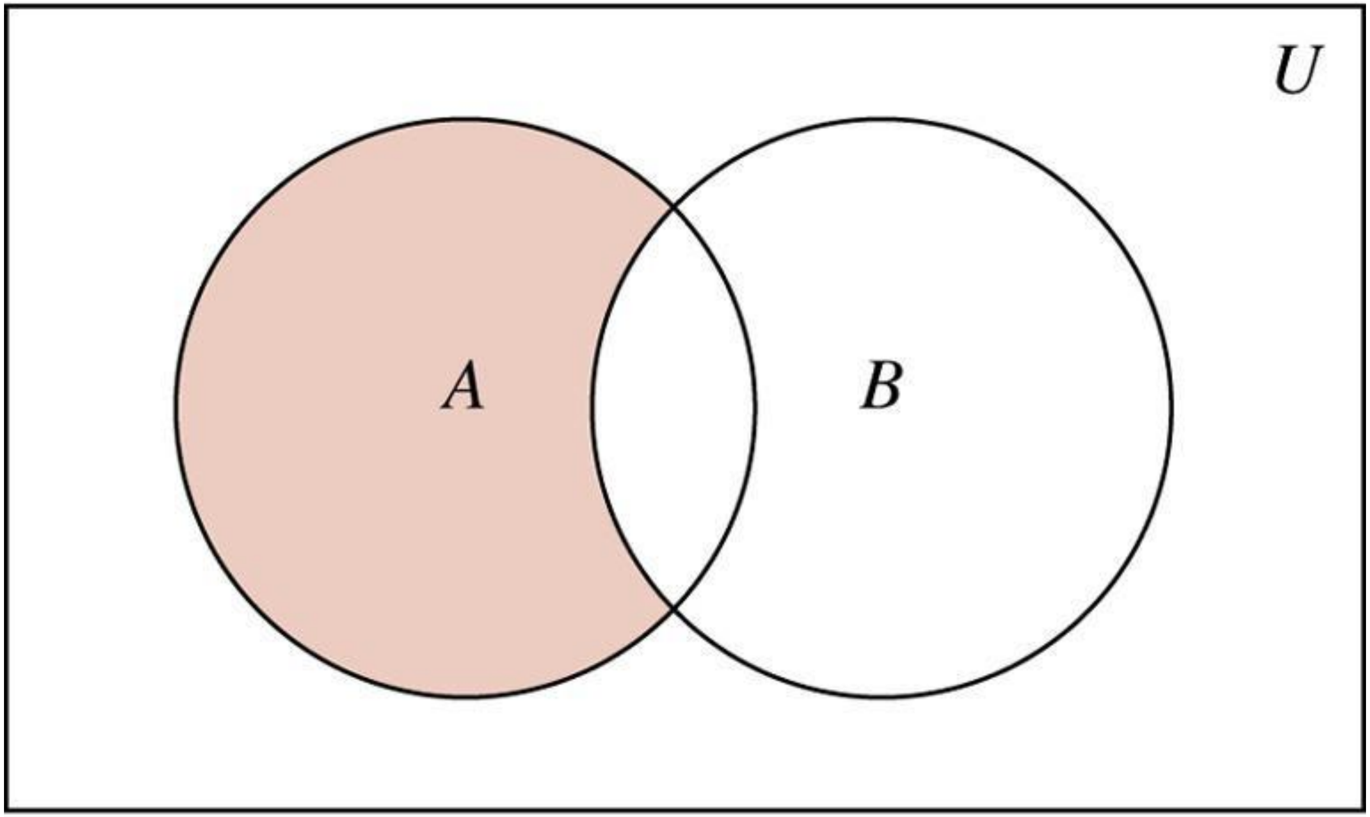


$A \cap B$ is shaded.

FIGURE 2 Venn Diagram Representing the Intersection of A and B.

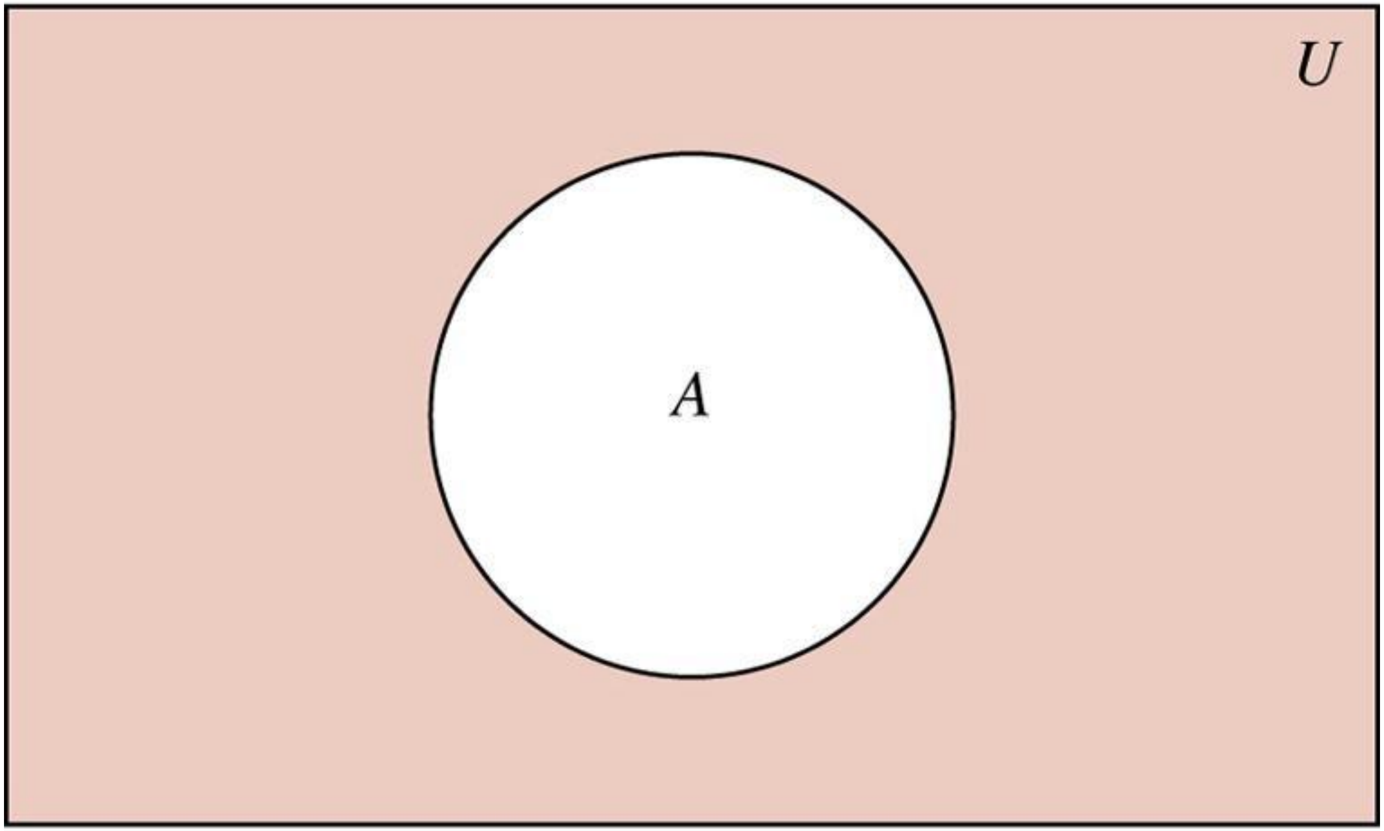
- **Definition 3:** Two sets are ***disjoint*** if their intersection is the empty set.
- $|A \cup B| = |A| + |B| - |A \cap B|$
  - **Principle of inclusion-exclusion**

- **Definition 4:** The ***difference*** of the sets A and B, denoted by A-B, is the set containing those elements that are in A but not in B.
  - Complement of B with respect to A
  - $A\text{-}B=\{x|x \in A \wedge x \notin B\}$
- **Definition 5:** The ***complement*** of the set A, denoted by Ā, is the complement of A with resepect to U.
  - $\bar{A} = \{x|x \notin A\}$



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B.



$\overline{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set *A*.

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

Set Identities

- To prove set identities
  - Show that each is a subset of the other
  - Using membership tables
  - Using those that we have already proved

TABLE 2 (2.2)

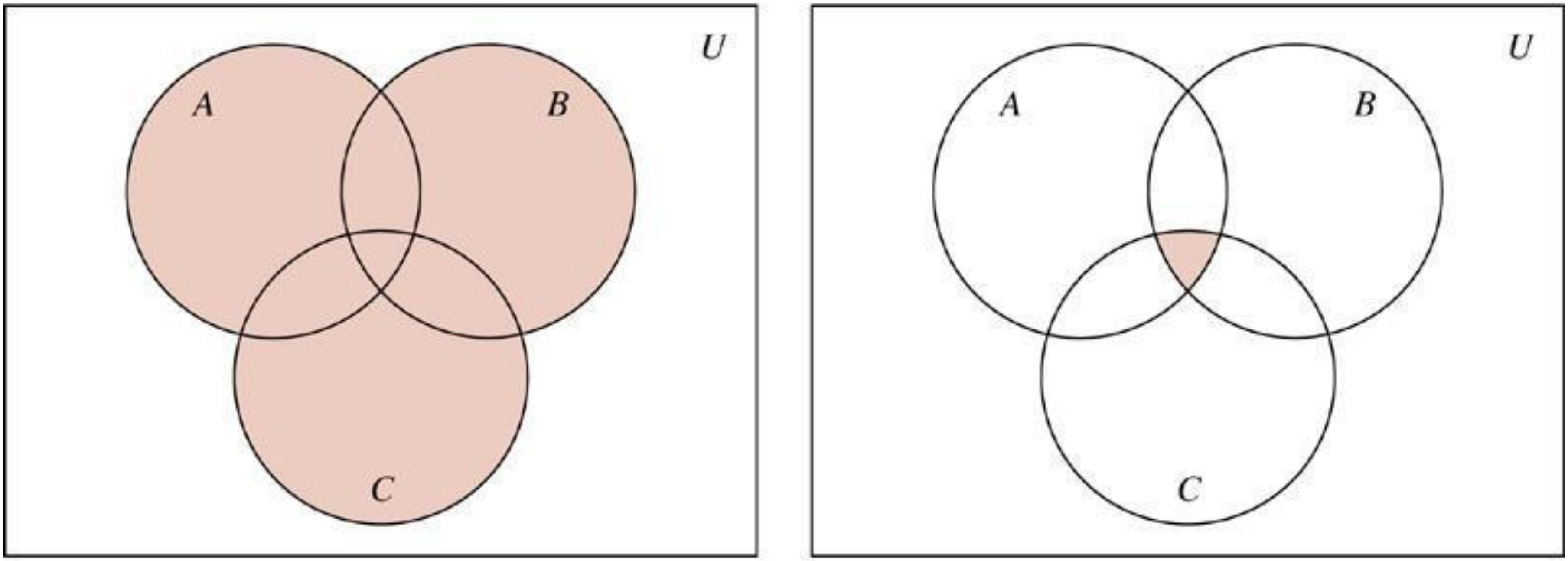
TABLE 2 **A Membership Table for the Distributive Property.**

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- **Definition 6:** The ***union*** of a collection of sets is the set containing those elements that are members of at least one set in the collection.
  - $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$
- **Definition 7:** The ***intersection*** of a collection of sets is the set containing those elements that are members of all the sets in the collection.
  - $A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$
- Computer Representation of Sets
  - Using bit strings

FIGURE 5 (2.2)



(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

FIGURE 5 The Union and Intersection of *A*, *B*, and *C*.

Getting Started

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2.3 Functions

- **Definition 1:** A ***function*** f from A to B is an assignment of exactly one element of B to each element of A. $f: A \to B$
- **Definition 2:** $f: A \to B$.
  - A: ***domain*** of f, B: ***codomain*** of f.
  - $f(a)=b$, a: ***preimage*** of b, b: ***image*** of a.
  - ***Range*** of f: the set of all images of elements of A
  - f: maps A to B

FIGURE 1

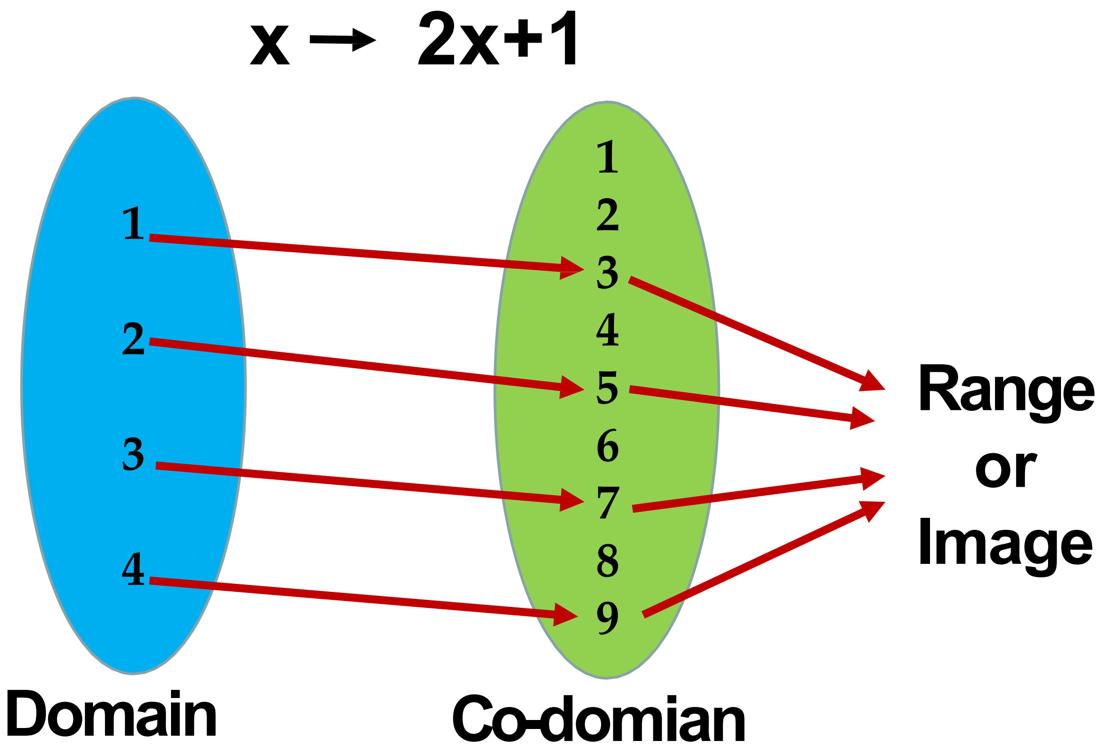

FIGURE 1.1: An example of function with it's components.

FIGURE 2

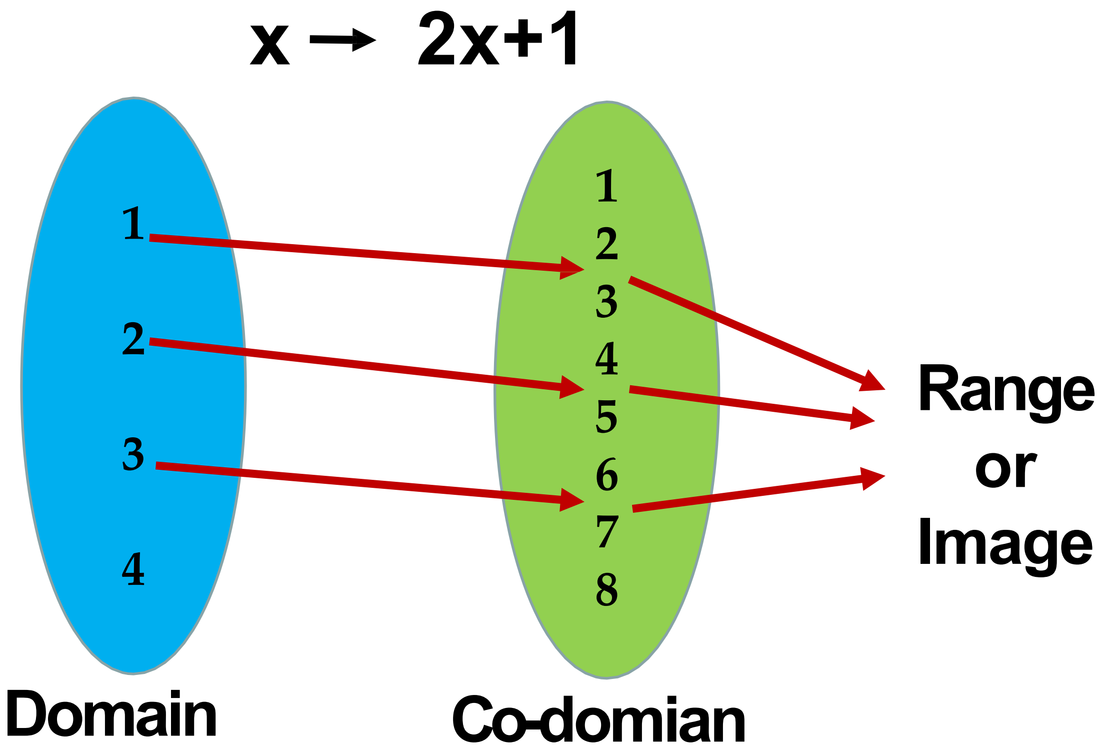

FIGURE 1.2: **An example of not being function**

FIGURE 3

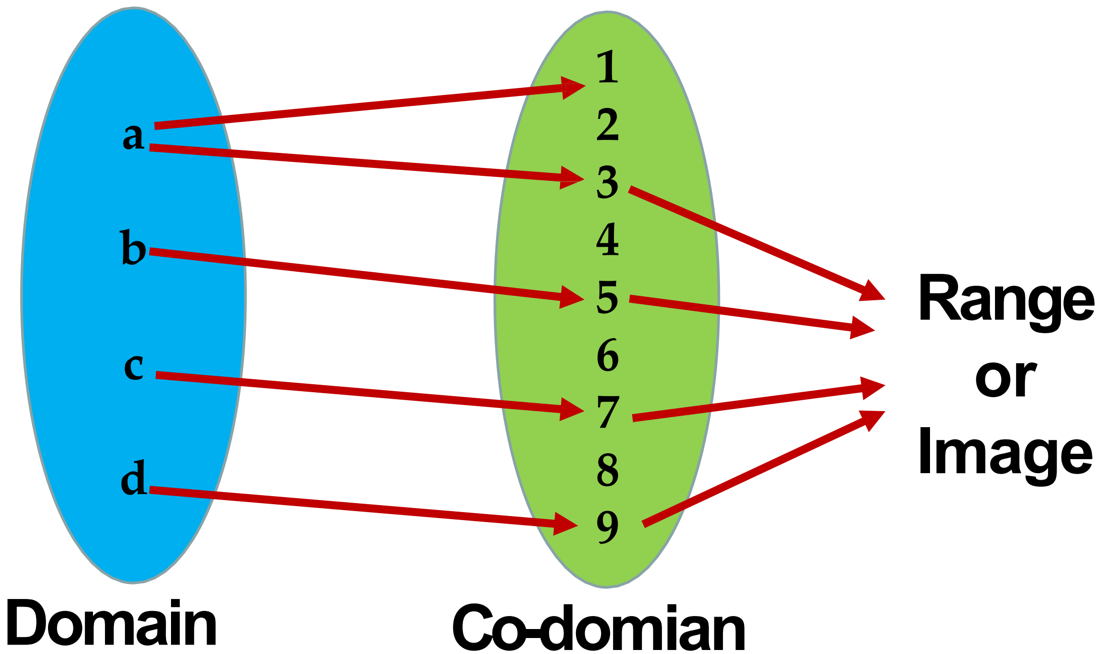

FIGURE 1.3: An example of not being function

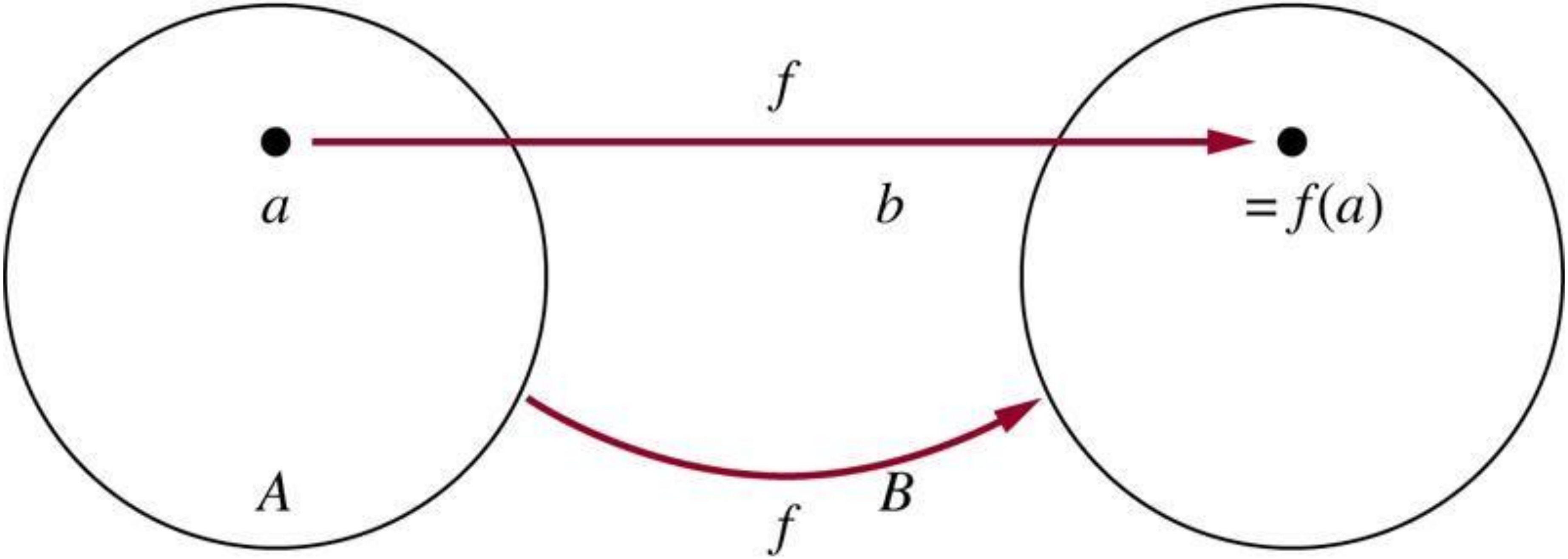

FIGURE 2 The Function f Maps *A* to *B*.

- **Definition 3:** Let f_1 and f_2 be functions from A to **R**. f_1+f_2 and f_1f_2 are also functions from A to **R**:
  - $(f_1+f_2)(x) = f_1(x)+f_2(x)$
  - $(f_1f_2)(x)=f_1(x)f_2(x)$
- **Definition 4:** $f: A\rightarrow B$, S is a subset of A. The image of S under the function f is: $f(S)=\{t|\exists s\in S(t=f(s))\}$

One-to-One and Onto Functions

- **Definition 5:** A function *f* is ***one-to-one*** **or** ***injective***, iff $f(a)=f(b)$ implies that $a=b$ for all *a* and *b* in the domain of *f*.
  - $\forall a \forall b(f(a)=f(b) \rightarrow a=b)$ or $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$
- **Definition 6:** A function f is ***increasing*** if $f(x) \leq f(y)$, and ***strictly increasing*** if $f(x)<f(y)$ whenever $x<y$. f is called ***decreasing*** if $f(x) \geq f(y)$, and ***strictly decreasing*** if $f(x)>f(y)$ whenever $x<y$.

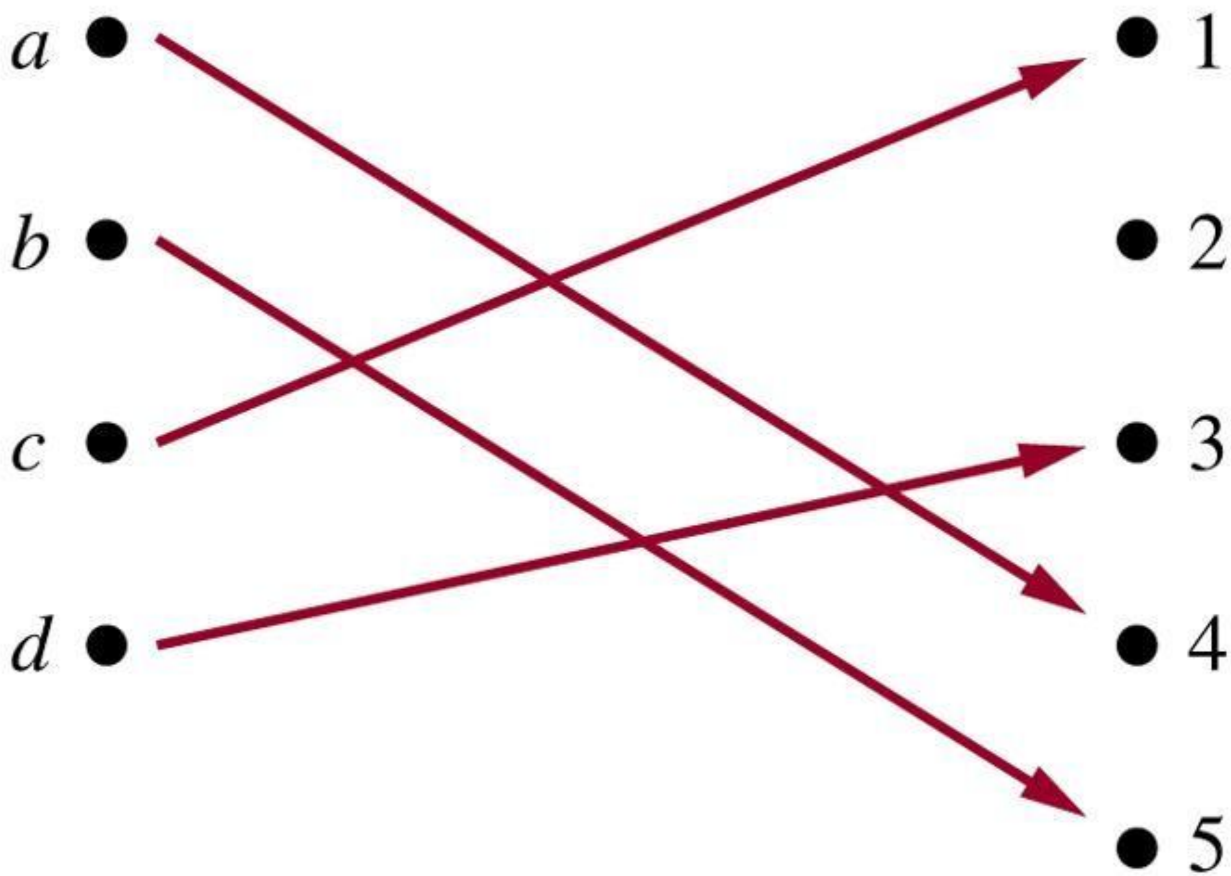

FIGURE 3 A One-to-One Function.

One-to-One and Onto Functions

- **Definition 7:** A function f is ***onto*** **or** ***surjective***, iff for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$.
  - $\forall y \exists x(\ f(x)=y\ )$ *or*
    $\forall a \forall b(\ a \neq b \rightarrow f(a) \neq f(b)\ )$
  - When ***co-domain = range***
- **Definition 8:** A function f is a ***one-to-one correspondence*** **or a** ***bijection*** if it is both one-to-one and onto.
  - Ex: identity function $\iota_A(x)=x$

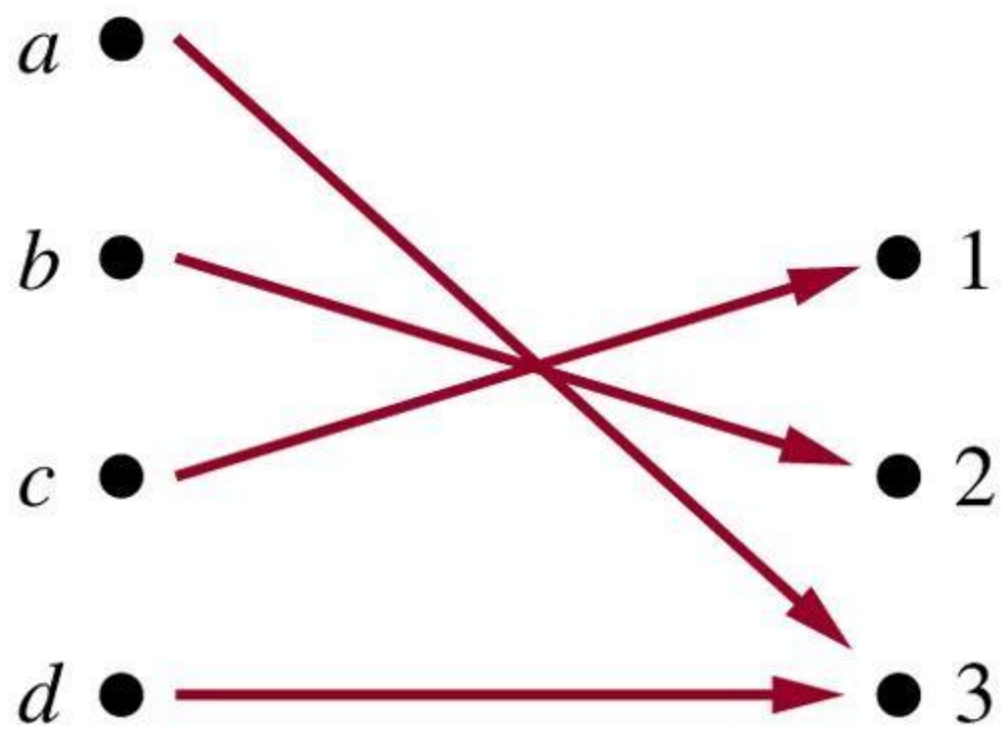

FIGURE 4 An Onto Function.

FIGURE 5 (2.3)

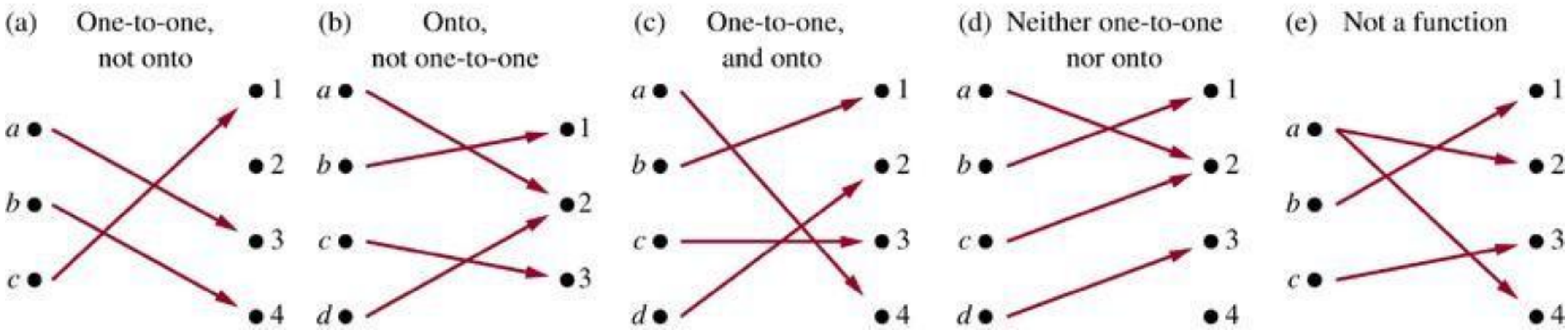

FIGURE 5 Examples of Different Types of Correspondences.

Inverse Functions and Compositions of Functions

- **Definition 9:** Let f be a one-to-one correspondence from A to B. The inverse function of f is the function that assigns to an element b in B the unique element a in A such that f(a)=b.
  - $f^{-1}(b)=a$ when $f(a)=b$

Prove It

↓

If a function is one-to-one correspondence then it's inverse is possible

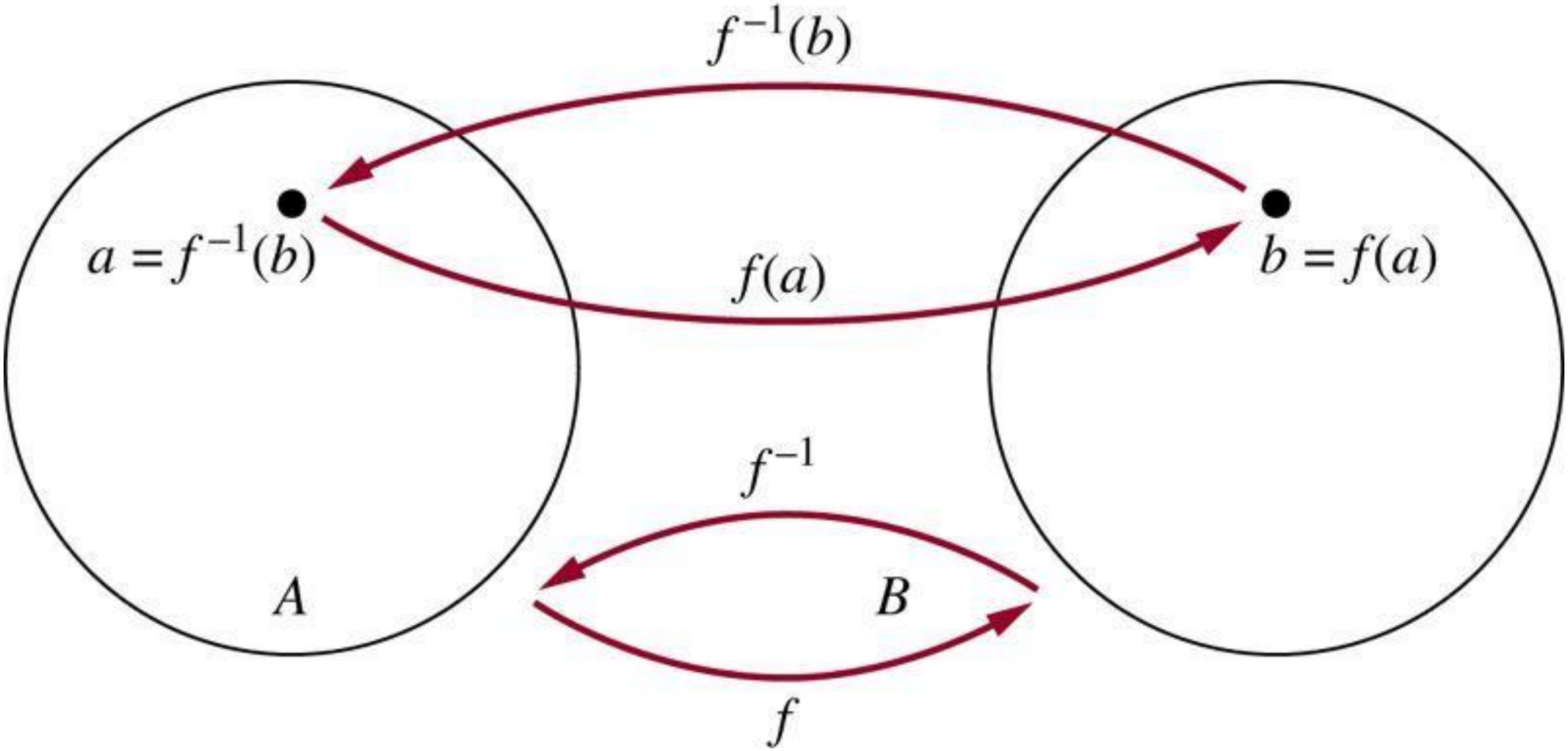

FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

- **Definition 10:** Let g be a function from A to B, and f be a function from B to C. The ***composition*** of functions f and g, denoted by $f \circ g$, is defined by:
  - $f \circ g(a) = f(g(a))$
  - $f \circ g$ and $g \circ f$ are not equal --- ***Prove it***

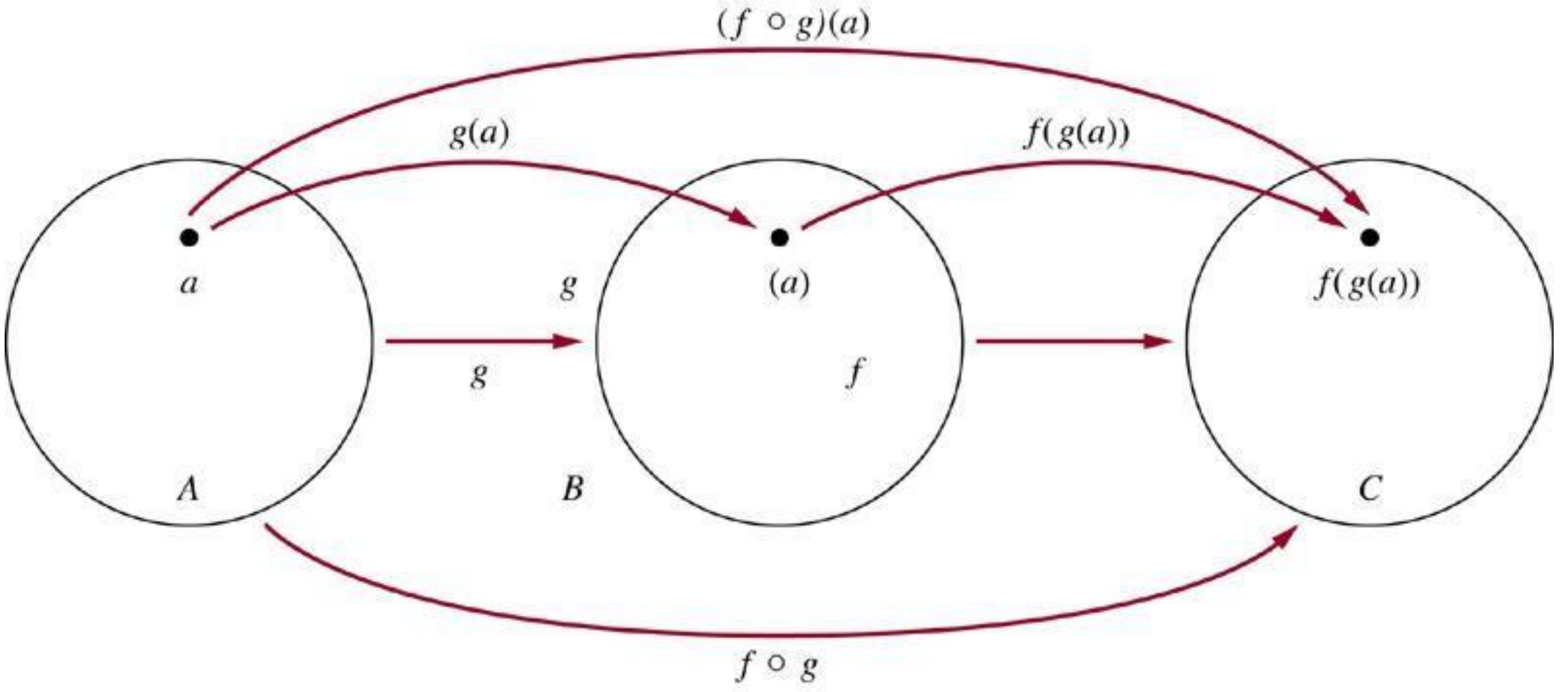

FIGURE 7 The Composition of the Functions *f* and *g*.

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

Graphs of Functions

- **Definition 11:** The ***graph*** of function f is the set of ordered pairs $\{(a,b) | a \in A \text{ and } f(a)=b\}$

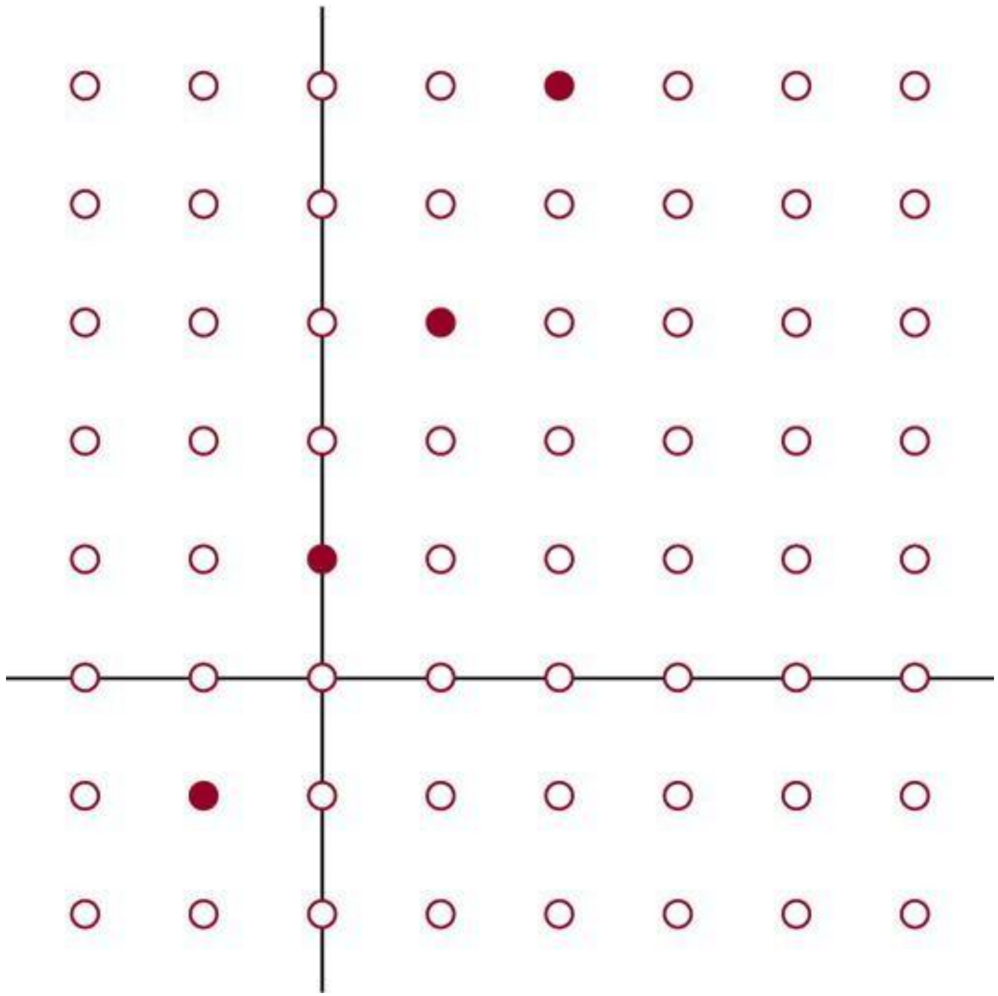

FIGURE 8 The Graph of $f(n) = 2n + 1$ from Z to Z.

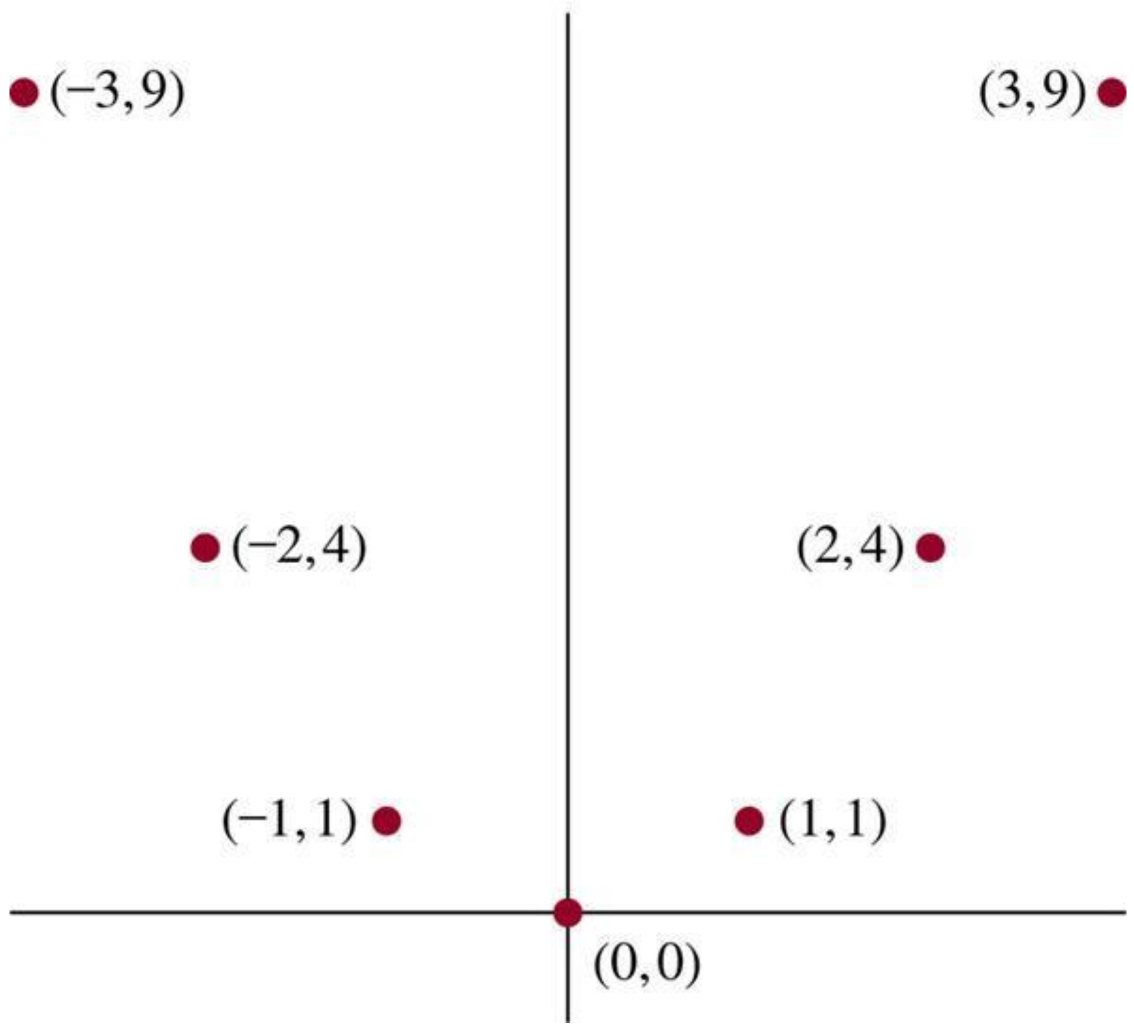

FIGURE 9 The Graph of $f(x) = x^2$ from Z to Z.

Floor and Ceil Functions

- **Definition 12:** The ***floor function*** assigns to *x* the largest integer that is less than or equal to *x* ($\lfloor x \rfloor$ or [*x*])..
- **Definition 13:** The ***ceiling function*** assigns to *x* the smallest integer that is greater than or equal to *x* ($\lceil x \rceil$)

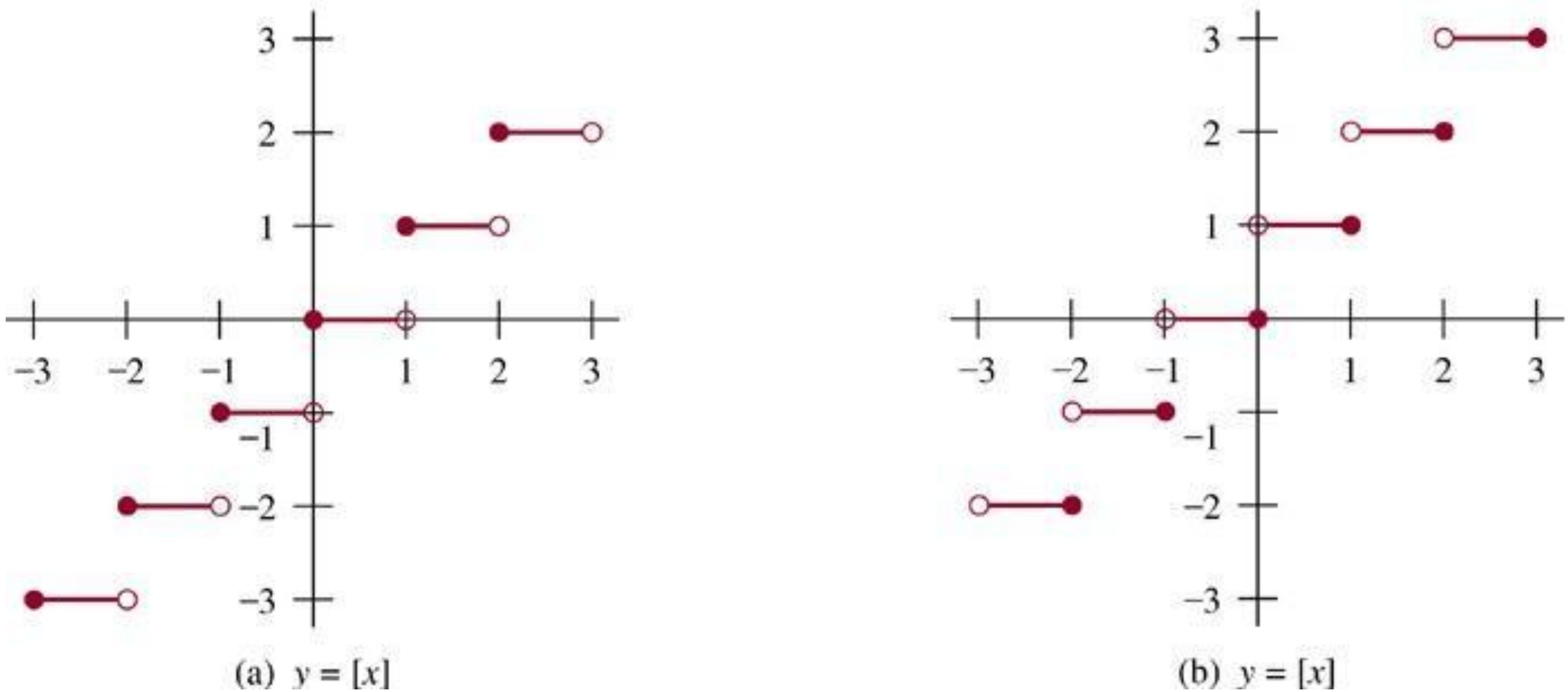

FIGURE 10 Graphs of the (a) Floor and (b) Ceiling Functions.

TABLE 1 (2.3)

TABLE 1 **Useful Properties of the Floor and Ceiling Functions.** (n is an integer, x is a real number)	
(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proof: 4(a)

- Let $\lfloor x \rfloor = m$ where m is a positive integer.
- By property 1(a)

$$m \leq x < m + 1$$

- Adding n on both sides

$$m + n \leq x + n < m + n + 1$$

- Using 1(a) again

$$\lfloor x + n \rfloor = m + n$$

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n$$

- A useful approach for considering statements about the floor function is to let

$$x = n + \varepsilon$$

- *where*

$$n = \lfloor x \rfloor,\ 0 \leq \varepsilon < 1$$

- For ceil

$$x = n - \varepsilon$$

- *where*

$$n = \lceil x \rceil,\ 0 \leq \varepsilon < 1$$

Proof: $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$

- Let $x = n + \varepsilon$ … … … … … … … … … … … … … … … … … ….*(1)*

  where $0 \le \varepsilon < 1$ and $n = \lfloor x \rfloor$ ……(c) is an integer

- Two cases to consider, depending on whether *ε is less than, or greater than or equal to* $\frac{1}{2}$.

❖ First let $0 \le \varepsilon < \frac{1}{2}$ … … … … … … … … … … … … … … … ….*(2)*

  or $0 \le 2\varepsilon < 1$

❖ From (1) $2x = 2n + 2\varepsilon$ … … … … … … … … … … … ….*(3)*

  and $\lfloor 2x \rfloor = 2n$

❖ Similarly from (1) $x + \frac{1}{2} = n + \frac{1}{2} + \varepsilon$ and $\lfloor 2x \rfloor = 2n$ … … *(a)*

❖ From (2) $\frac{1}{2} \le \varepsilon + \frac{1}{2} < 1$ or $0 < \varepsilon + \frac{1}{2} < 1$

  and so $\left\lfloor x + \frac{1}{2} \right\rfloor = n$ … ….*(b)*

❖ From (a) and (b)+(c), $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + n = 2n$

  So $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$

Proof: $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$

- Let $\frac{1}{2} \leq \varepsilon < 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (4)$

  or $1 \leq 2\varepsilon < 2$ or $0 \leq 2\varepsilon - 1 < 1$
- From (3) $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$

  and $\lfloor 2x \rfloor = 2n + 1 \ldots. (x)$
- Similarly from (1) $x + \frac{1}{2} = n + \frac{1}{2} + \varepsilon = (n + 1) + (\varepsilon - \frac{1}{2}) \ldots\ldots (y)$
- From (4) $0 \leq \varepsilon - \frac{1}{2} < \frac{1}{2}$ or $0 < \varepsilon - \frac{1}{2} < 1$

  and so $\left\lfloor x + \frac{1}{2} \right\rfloor = n + 1 \ldots\ldots. (b)$
- From (x) and (y)+(c), $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + n + 1 = 2n + 1$

So $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$

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2.4 Sequences and Summations

- **Definition 1:** A ***sequence*** is a function from a subset of the set of integers to a set S. We use a_n to denote the image of the integer n (a *term* of the sequence)
  - The sequence $\{a_n\}$
    - Ex: $a_n=1/n$

- **Definition 2:** A ***geometric progression*** is a sequence of the form

  $a, ar, ar^2, \ldots, ar^n, \ldots$

  where the *initial term a* and the *common ratio r* are real numbers
- **Definition 3:** A ***arithmetic progression*** is a sequence of the form

  $a, a+d, a+2d, \ldots, a+nd, \ldots$

  where the *initial term a* and the *common difference d* are real numbers

- Ex. 1, 3, 5, 7, 9, …
- Ex. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …
- Ex. 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047, …

TABLE 1 (2.4)

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .

Summing a Sequence

- Specific sum

$$1 + 2 + 3 + \ldots\ldots + n\text{-}1 + n$$

- General summation of a sequence of *terms:*

$$a_1 + a_2 + \ldots \ldots \; a_n$$

$$a_k \;\; \textit{terms}$$

$$\textit{where } k = 1 , 2 , \ldots \;\; n$$

Summing a Sequence

- Each element a_k of a sum is called a ***term***.
- The terms are often specified ***implicitly as formulas that follow a readily perceived pattern.***
- In such cases we must sometimes write them in an expanded form so that the meaning is clear.

Summing a Sequence

$$1 + 2 + \dots\dots\dots.$$

$$1 + 2 + \dots.. + 2^{n-1}$$

$$1 + 2 + 4 + \dots\dots. + 2^{n-1}$$

$$2^0 + 2^1 + 2^2 + \dots\dots. + 2^{n-1}$$

The three-dots notation has many uses, but it can be ambiguous and a bit long-winded

Delimited Form of Sum

- Three-dots notation is vague and wordy

$$\sum_{k=1}^{n} a_k$$

which is also called ***Sigma-notation*** because it uses the Greek letter $\sum$

Delimited Form of Sum

- Parts of notation
  - Summand
  - Index variable
  - Lower limit
  - Upper limit

Delimited Form of Sum

- Sigma notation inline

$$\textstyle\sum_{k=1}^{n} k$$

Delimited Form of Sum

$$\sum_{k=1}^{n} a_k$$

Sums the terms $\boldsymbol{a_k}$ *where index* ***k*** *is an integer*

from lower limit ***1*** *to upper limit* ***n***

or

sum over ***k*** *from* ***1*** *to* ***n***

Generalized Sigma-Notation

- Specify a condition that the index variable must satisfy

$$\sum_{1 \leq k \leq n} a_k$$

- We simply write one or more conditions under the $\sum$ to specify the set of indices over which summation should take place.

Generalized Sigma-Notation

- The general form allows us to take sums over index sets that aren’t restricted to consecutive integers.

Generalized Sigma-Notation

- Express the sum of the squares of all odd positive integers below 100

$$\sum_{\substack{1 \leqslant k < 100 \\ k \text{ odd}}} k^2$$

- The delimited equivalent of this sum

$$\sum_{k=0}^{49} (2k+1)^2$$

Generalized Sigma-Notation

- The sum of reciprocals of all prime numbers between 1 and N

$$\sum_{\substack{p \leqslant N \\ p \quad \text{prime}}} \frac{1}{p}$$

- The delimited equivalent of this sum

$$\sum_{k=1}^{\pi(N)} \frac{1}{p_k}$$

π(N) = Number of primes given

Advantage of Generalized Sigma-Notation

- We can manipulate it more easily than the delimited form

Advantage of Generalized Sigma-Notation

- Change the index variable ***k*** to ***k + 1***
- Generalized Sigma-notation

$$\sum_{1 \leqslant k \leqslant n} a_k = \sum_{1 \leqslant k+1 \leqslant n} a_{k+1}$$

- Delimited form

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1}$$

Advantage of Delimited Form

- It's nice and tidy, and we can write it quickly

$$\sum_{k=1}^{n} a_k = \sum_{1 \leqslant k \leqslant n} a_k$$

- Needs less symbol than generalized sigma notation.

Delimited Form Vs. Generalized Sigma-Notation

- ***Delimited Form***
  - Used in presenting or stating a problem
- ***Generalized Sigma-Notation***
  - Used when index variable needs to be transformed.

Summations

- Summation notation:

$$\sum_{j=m}^{n} a_j \qquad \sum\nolimits_{j=m}^{n} a_j \qquad \sum\nolimits_{1 \le j \le n} a_j$$

  - $a_m + a_{m+1} + \ldots + a_n$
  - j: index of summation
  - m: lower limit
  - n: upper limit

- Theorem 1 (*geometric series*): If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

  - ***Prove it yourself***

Proof: Let

$$S_n = \sum_{j=0}^{n} ar^j.$$

To compute S, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned} rS_n &= r\sum_{j=0}^{n} ar^j && \text{substituting summation formula for } S \\ &= \sum_{j=0}^{n} ar^{j+1} && \text{by the distributive property} \\ &= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j+1 \\ &= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a) && \text{removing } k = n+1 \text{ term and adding } k = 0 \text{ term} \\ &= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula} \end{aligned}$$

From these equalities, we see that

$$rS_n = S_n + (ar^{n+1} - a).$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a.$ ◁

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1 - x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1 - x)^2}$

Cardinality

- **Definition 4:** The sets A and B have the ***same cardinality*** iff there is a one-to-one correspondence from A to B.
- **Definition 5:** A set that is either finite or has the same cardinality as the set of positive integers is called ***countable***. A set that is not countable is called ***uncountable***. When an infinite set S is countable, $|S| = \aleph_0$ ("aleph null")
  - Ex: the set of odd positive integers is countable

FIGURE 1 (2.4)

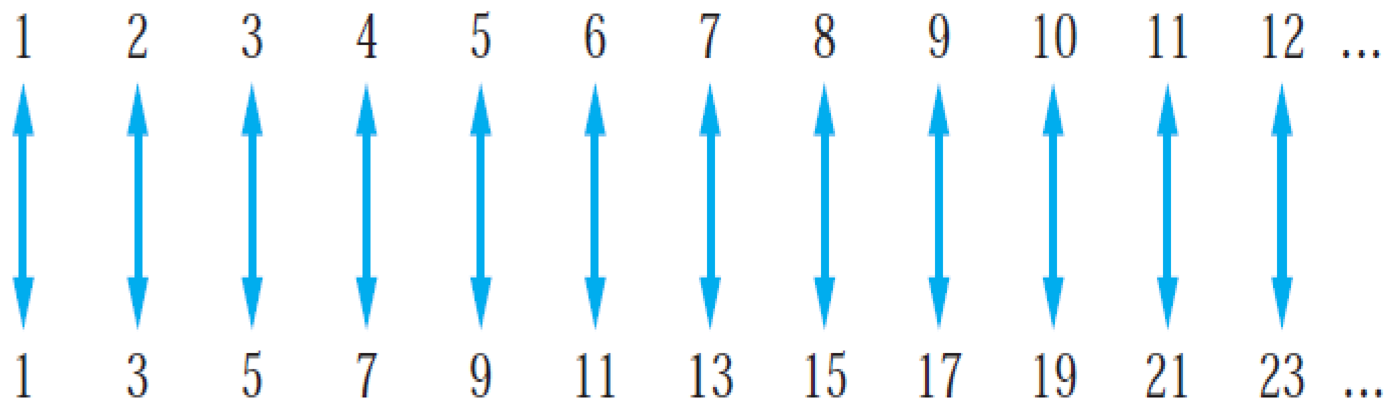

FIGURE 1 A One-to-One Correspondence Between $\mathbf{Z}^+$ and the Set of Odd Positive Integers, $f(n) = 2n - 1$.

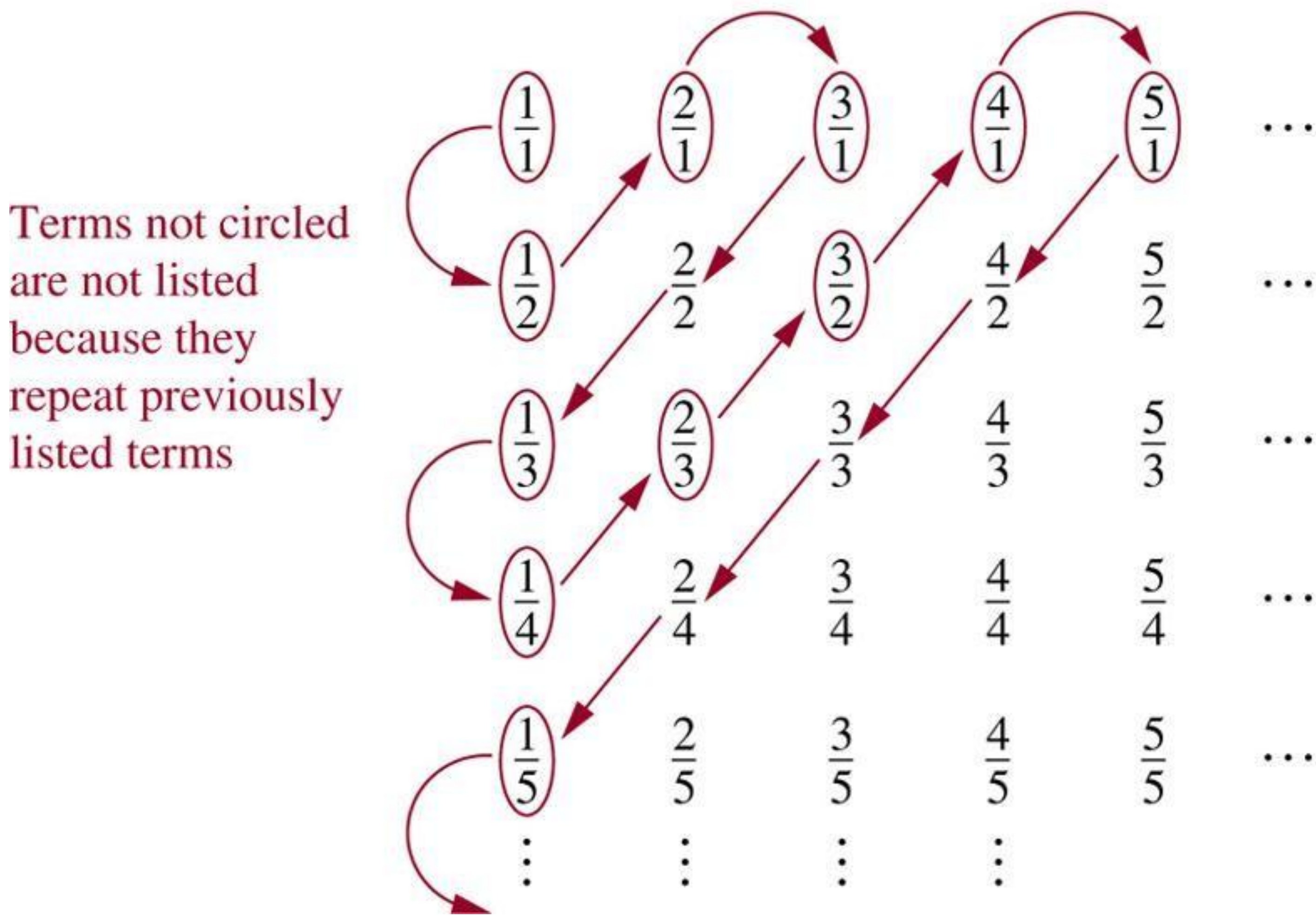

FIGURE 2 The Positive Rational Numbers Are Countable.

Practice Problem

- Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if $f(x) = x/5$.

Apply $f(x)=x/5$ to each $x \in S$:

$f(-1)=-1/5=-0.2$

$f(0)=0/5=0$

$f(2)=2/5=0.4$

$f(4)=4/5=0.8$

$f(7)=7/5=1.4$

If x is a real number and m is an integer, then prove that $\lceil x+m \rceil = \lceil x \rceil + m$.

Express x in terms of its ceiling:

Let $\lceil x \rceil = n$, where n is the smallest integer greater than or equal to x.

By definition of the ceiling function:

$n-1 < x \le n$.

Add the integer m*m* to all parts of the inequality:

$(n-1)+m < x+m \le n+m$.

Simplifying:

$(n+m)-1 < x+m \le n+m$.

Apply the ceiling function to x+m:

The inequality $(n+m)-1 < x+m \le n+m$ shows that x+m lies in the interval between $(n+m)-1$ and $n+m$.

By definition, the ceiling of x+m is the smallest integer greater than or equal to x+m, which is n+m:

$\lceil x+m \rceil = n+m$.

Substitute $\lceil x \rceil = n$:

$\lceil x+m \rceil = \lceil x \rceil + m$.